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Cooperation and Coordination in Threshold Public Goods Games with Asymmetric Players

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Abstract: In this paper, we study cooperation and coordination in a threshold public goods game with asymmetric players where players have different endowments e_i, productivities p_i, and rewards r_i. In general, this game has a defective Nash equilibrium (NE), in which no one contributes, and multiple cooperative NEs, in which the group's collective contribution equals the threshold. We then study how multiple dimensions of inequality influence people's cooperation and coordination. We show that heterogeneity in $e_i p_i$ can promote cooperation in the sense that the existence condition of the defective NE becomes stricter. Furthermore, players with higher $e_i p_i$ are likely to contribute more at a cooperative NE in terms of collective contribution (i.e., absolute contribution multiplied by productivity).

Keywords: threshold public goods game; asymmetric game; cooperation; coordination

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1. Introduction

Exploring cooperation and coordination stands out as a fundamental application of game theory. This field seeks to elucidate which mechanisms drive cooperative behavior and how individuals coordinate towards a more efficient equilibrium. Much of the existing work in this field considers homogeneous populations. However, inequality is ubiquitous in humans. Individuals often vary in their endowments, productivities, shares of rewards, and social hierarchy positions. Asymmetric games raise many exciting questions that are difficult to tackle within the standard setup of symmetric games. For example, what is the impact of various forms of inequality on cooperation and coordination? Does endowment inequality make cooperation and coordination more difficult than productivity inequality? How do individuals coordinate when they differ in multiple dimensions?

The existing literature suggests that the effect of inequality on cooperation is nontrivial. A considerable number of studies based on public goods games has found that endowment inequality tends to reduce cooperation [1–3] but that asymmetric productivities and the sharing of public goods tend to positively influence contributions or maintain a neutral impact [4–7]. In addition, in cases of both endowment and productivity heterogeneity, optimal cooperation is achieved when individuals with a higher productivity also possess greater endowments [8].

In contrast, the interaction between inequality and coordination has attracted less attention. In this paper, we discuss this issue within the framework of a simple coordination game, a threshold public goods game (TPGG). In a TPGG, individuals derive benefits from contributing to a public good only when their total collective contributions (i.e., absolute contribution multiplied by productivity) reach a certain threshold [9–11]. In general, a threshold public goods game tends to have many equilibria, including a defective equilibrium in which no contributions are made and a set of cooperative equilibria in which the group's collective contribution exactly meets the threshold [9,11]. A cooperative equilibrium is more efficient than a defective equilibrium; therefore, everyone has the motivation to achieve the threshold. However, different players may prefer different cooperative equilibria. They may prefer other group members to contribute more, allowing them to contribute less, thereby leading to a nontrivial coordination problem.

When the TPGG is symmetric, i.e., all players have the same endowment, productivity, and share of the reward, a natural solution is that all the players contribute the same amount of their endowment to the public good. However, when the TPGG is asymmetric, experimental studies have demonstrated a negative effect of endowment inequality on coordination[1]: while rich subjects often make higher contributions compared to the symmetric case, the much less contributions from the poor lead to an overall decrease in the total collective contribution in the asymmetric scenario [11–13]. In contrast, it seems that productivity inequality does not have a significant effect [11]. The effect of inequality on cooperation and coordination has also been investigated in a variation of the TPGG, the climate game (also known as the collective-risk dilemma game) [14]. Experiments have found that achieving coordination is hindered more by endowment inequality than by productivity inequality [15–18]. Furthermore, the effects of multiple inequalities on coordination are generally more complex [19].

Most of the existing studies on asymmetric TPGGs have considered two types of players with only one source of asymmetry: they either differ in endowments or in productivities[2]. It is not clear how multiple dimensions of inequality influence people's cooperation and coordination in large groups. In this paper, we consider an n-player asymmetric TPGG where players have different endowments, productivities, and rewards. We focus on the following two questions:

- What is the impact of various forms of inequality on cooperation?
- How do people coordinate when group members differ among multiple dimensions?

We investigate the impact of various forms of inequality on cooperation by examining the existence conditions of Nash equilibria. Furthermore, to analyze the coordination among players with multidimensional differences, we quantify the size of cooperative NE subset in which a specific player makes the largest relative, absolute, or collective contribution.

Section 2 provides a detailed description of the asymmetric TPGG model. In Section 3, two theorems are presented to address the two key questions. Section 4 conducts a numerical analysis, illustrating the theoretical findings with concrete examples. Conclusive remarks are given in Section 5.

2. Model

Let us consider a threshold public goods game (TPGG) with n asymmetric players. Each player i (with $i = 1, \dots, n$) starts with some endowment, meaning $e_i > 0$. Then, each player independently decides which fraction x_i of their endowment to contribute to the public good. Each player's contribution is multiplied by a productivity factor p_i. We refer to x_i, $e_i x_i$, and $e_i p_i x_i$ as player i's relative contribution, absolute contribution, and collective contribution, respectively [11]. If the total collective contribution of the n players reaches a predefined threshold θ, then each player i obtains a reward r_i along with their remaining endowment. Otherwise, the players only receive their remaining endowment. As a result, the players' payoffs are as follows:

$$f_i(\boldsymbol{x}) = \begin{cases} e_i(1 - x_i) + r_i, & \text{if } \sum_{i=1}^{n} e_i p_i x_i \geq \theta, \\ e_i(1 - x_i), & \text{if } \sum_{i=1}^{n} e_i p_i x_i < \theta. \end{cases}$$

This game can have various kinds of heterogeneities with respect to endowment e_i, productivity p_i, and reward r_i. To compare the impact of various forms of inequality, we assume $\sum_{i=1}^{n} e_i p_i = G$ and $\sum_{i=1}^{n} r_i = R$, i.e., the maximum amount of group collective contribution and the total amount of reward are fixed. Furthermore, we assume $r_i \geq e_i$ for all $i = 1, 2, \ldots, n$, i.e., the social welfare is improved if the group reaches the threshold[3].

3. Results

In general, an asymmetric TPGG has two types of NE. First, there is a set of cooperative NEs in which the total collective contribution exactly meets the threshold. In addition, there may exist a defective NE in which all the players choose not to contribute. Theorem 1 provides the existence conditions for the two types of NE (see Appendix A for the proof).

Theorem 1. *The defective NE, $\boldsymbol{x} = \mathbf{0}$, exists if and only if, for any player i, (1) $\theta > e_i p_i$ or (2) $\theta = e_i p_i$ and $r_i = e_i$. In contrast, a cooperative NE (i.e., at least one player contributes to the public good) exists if and only if $0 < \theta \leq G$. In addition, the set of cooperative NEs has the form $\{\boldsymbol{x} | \sum_{i=1}^{n} e_i p_i x_i = \theta\}$.*

Theorem 1 reveals the impact of various inequalities on cooperation from the perspective of NE. It indicates that individual differences have no effect on the existence of a cooperative NE. The cooperative NE set is not empty if and only if the maximum amount of group collective contribution is not less than the threshold. We then analyze the effect of heterogeneity on the defective NE by assuming $r_i > e_i$ (i.e., the critical case $r_i = e_i$ is excluded). For the symmetric case, the existence condition of the defective NE can be simplified as $\theta > \frac{G}{n}$. For the asymmetric case, the defective NE exists if and only if $\theta > e_i p_i$ for all players. Without a loss of generality, we consider that players differ in $e_i p_i$, with $e_1 p_1 > e_2 p_2 \ldots > e_n p_n$ (i.e., player 1 is the most 'able' player). In this case, the defective NE exists if and only if $\theta > e_1 p_1$. Let us note that $e_1 p_1 > \frac{\sum_{i=1}^{n} e_i p_i}{n} = \frac{G}{n}$, and the existence condition of the defective NE becomes stricter than that in the symmetric case. This implies that heterogeneity in $e_i p_i$ has a positive impact on cooperation. In particular, defection ceases to be an NE when $e_1 p_1 \geq \theta$ because, in this case, player 1 is motivated to contribute even if all the other players defect.

Theorem 1 also shows that, when $0 < \theta < G$, the game has an infinite number of cooperative NEs. In all these equilibria, the group collective contribution reaches the threshold (i.e., $\sum_{k=1}^{n} e_k p_k x_k = \theta$); yet, they vary in how the contributions are distributed among the players. We then analyze which player contributes more to a cooperative NE in the presence of asymmetries. We note that there are three measures of individual contribution, namely, relative contribution, absolute contribution, and collective contribution. We denote the subsets of the cooperative NE set where player i has the highest relative contribution, absolute contribution, and collective contribution by S_i^1, S_i^2, and S_i^3, respectively, and we denote their sizes (under the Hausdorff measure) by $|S_i^1|$, $|S_i^2|$, and $|S_i^3|$, respectively. Intuitively, $\left|S_i^k\right| > \left|S_j^k\right|$ for all $j \neq i$ ($k = 1, 2, 3$) means that player i is more likely to contribute most at a randomly chosen cooperative NE in terms of the relative, absolute, or collective contribution[4]. Theorem 2 compares the sizes of the different subsets (see Appendix B for the proof).

Theorem 2. *Let us suppose that $e_1 p_1 > e_2 p_2 \ldots > e_n p_n$.*

(1) *For relative contribution, $|S_1^1| < \ldots < |S_n^1|$ if $0 < \theta < e_n p_n$, and $|S_1^1| > \ldots > |S_n^1|$ if $\sum_{i=1}^{n-1} e_i p_i < \theta < G$.*
(2) *For absolute contribution, when there is endowment heterogeneity, i.e., $e_1 > \ldots > e_n$ and $p_1 = \ldots = p_n$, $|S_1^2| \geq \ldots \geq |S_n^2|$ for all $0 < \theta < G$. When there is productivity heterogeneity, i.e., $e_1 = \ldots = e_n$ and $p_1 > \ldots > p_n$, $|S_1^2| < \ldots < |S_n^2|$ if $0 < \theta < e_n p_n$, and $|S_1^2| > \ldots > |S_n^2|$ if $\sum_{i=1}^{n-1} e_i p_i < \theta < G$.*
(3) *For collective contribution, $|S_1^3| \geq \ldots \geq |S_n^3|$ for all $0 < \theta < G$. Furthermore, $|S_1^3| = \ldots = |S_n^3|$ if $0 < \theta < e_n p_n$.*

Theorem 2 summarizes the contribution patterns in cooperative NEs when group members differ across multiple dimensions. First, 'able' players (i.e., players with larger $e_i p_i$) are likely to make higher collective contributions in the sense that $\left|S_i^3\right| \geq \left|S_j^3\right|$ if $e_i p_i > e_j p_j$. In addition, if the threshold is sufficiently low such that any single player is able to achieve it, then all players are equally likely to make the highest collective contribution in a cooperative NE, irrespective of their abilities. Second, 'able' players are likely to have higher relative contributions only for large threshold, where successfully reaching the threshold requires the joint efforts of all players. However, for sufficiently low threshold, 'able' players tend to make smaller relative contributions. Finally, the results for absolute contribution depend on both e_i and p_i. When there is endowment inequality, the players with higher endowments are likely to contribute more for all thresholds. However, when there is productivity inequality, the more productive players are likely to contribute more only for large threshold.

4. Numerical Analysis

Theorems 1 and 2 qualitatively analyze the effect of various forms of inequality on cooperation and coordination. In particular, Theorem 2 compares the relative sizes of S_i^k for low thresholds (i.e., $0 < \theta < e_n p_n$) and high thresholds (i.e., $\sum_{i=1}^{n-1} e_i p_i < \theta < G$). In order to enhance the comprehension of the two theorems, we calculate the Nash equilibria and absolute sizes of the three types of subsets S_i^1, S_i^2, and S_i^3 for two-player threshold public goods games with all $0 < \theta < G$. Inspired by [8], we consider five scenarios, namely, full equality, endowment inequality, productivity inequality, aligned inequality, and misaligned inequality. Full equality corresponds to a homogeneous scenario, whereas in the endowment inequality and productivity inequality scenarios players only differ in a single dimension. In the last two scenarios, players differ in two dimensions: in the aligned inequality scenario, both advantages concentrate in one player, with the high-endowment player being more productive, while in the misaligned inequality scenario these two advantages distribute across two players, meaning the high-endowment player is less productive. Furthermore, $e_1 p_1 = e_2 p_2 = 30$ in the full equality and misaligned inequality scenarios; $e_1 p_1 = 40, e_2 p_2 = 20$ in the endowment inequality and productivity inequality scenarios, and $e_1 p_1 = 48, e_2 p_2 = 12$ in the aligned inequality scenario. Detailed parameter settings for e_i and p_j can be found in the first column of Table 1 ($G = 60$ and $r_1 = r_2 = 45$ in all five scenarios). For each scenario, the threshold θ is categorized into three ranges according to Theorem 2, i.e., low thresholds $0 < \theta < e_2 p_2$, intermediate thresholds $e_2 p_2 \leq \theta \leq e_1 p_1$, and high thresholds $e_1 p_1 < \theta < G$.

We first study the effect of inequality on cooperation. Theorem 1 says that heterogeneity in $e_i p_i$ can promote cooperation in the sense that the existence condition of the defective NE becomes stricter. From Table 1, the existence conditions of the defective NE in the full equality, endowment inequality, productivity inequality, aligned inequality, and misaligned inequality scenarios are $\theta > 30$, $\theta > 40$, $\theta > 40$, $\theta > 48$, and $\theta > 30$, respectively. This shows clearly that the existence condition of the defective NE is harder to satisfy when heterogeneity in $e_i p_i$ increases.

We then analyze the set of cooperative NEs. Figure 1 shows the set of cooperative NEs for different game scenarios and thresholds θ. For a two-player threshold public goods game, the set of cooperative NEs is a line segment in the x_1-x_2 plane with slope $-\frac{e_1 p_1}{e_2 p_2}$. Furthermore, the length of the line segment (which measures the size of the cooperative NE set) does not change monotonically in the threshold: it first increases and then decreases as θ increases from 0 to G.

Table 1. Nash equilibria and sizes of S_i^k in two-player threshold public goods games. Column 1: Five game scenarios are considered, namely, full equality, endowment inequality, productivity inequality, aligned inequality, and misaligned inequality. Column 2: For each scenario, the threshold θ is categorized into three ranges according to Theorem 2. Column 3: The cooperative NE set is non-empty for all parameter combinations, and the defective NE exists only for high thresholds. Columns 4–6: Absolute sizes of the three sets S_i^1, S_i^2, and S_i^3. The size relationship between $\left|S_1^k\right|$ and $\left|S_2^k\right|$ is marked using different colors. Red: $\left|S_1^k\right| > \left|S_2^k\right|$. Black: $\left|S_1^k\right| = \left|S_2^k\right|$. Blue: $\left|S_1^k\right| < \left|S_2^k\right|$. Grey: the relationship between $\left|S_1^k\right|$ and $\left|S_2^k\right|$ depends on θ. Three cells are highlighted. In these cells, $\left|S_1^3\right| > \left|S_2^3\right|$ for almost all θ. The exceptions are $\theta = 20$ in the first two cells and $\theta = 12$ in the last cell.

(e_1,e_2,p_1,p_2)	**Threshold**	**NE**	$(\lvert S_1^1\rvert,\lvert S_2^1\rvert)$	$(\lvert S_1^2\rvert,\lvert S_2^2\rvert)$	$(\lvert S_1^3\rvert,\lvert S_2^3\rvert)$
Full equality $(30,30,1,1)$	$0<\theta<30$	$x_1+x_2=\frac{\theta}{30}$	$\left(\frac{\sqrt{2}\theta}{60},\frac{\sqrt{2}\theta}{60}\right)$	$\left(\frac{\sqrt{2}\theta}{60},\frac{\sqrt{2}\theta}{60}\right)$	$\left(\frac{\sqrt{2}\theta}{60},\frac{\sqrt{2}\theta}{60}\right)$
	$\theta=30$	$x_1+x_2=1$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
	$30<\theta<60$	$x_1+x_2=\frac{\theta}{30}$; $x_1=x_2=0$	$\left(\sqrt{2}-\frac{\sqrt{2}\theta}{60},\sqrt{2}-\frac{\sqrt{2}\theta}{60}\right)$	$\left(\sqrt{2}-\frac{\sqrt{2}\theta}{60},\sqrt{2}-\frac{\sqrt{2}\theta}{60}\right)$	$\left(\sqrt{2}-\frac{\sqrt{2}\theta}{60},\sqrt{2}-\frac{\sqrt{2}\theta}{60}\right)$
Endowment inequality $(40,20,1,1)$	$0<\theta<20$	$2x_1+x_2=\frac{\theta}{20}$	$\left(\frac{\sqrt{5}\theta}{120},\frac{\sqrt{5}\theta}{60}\right)$	$\left(\frac{\sqrt{5}\theta}{80},\frac{\sqrt{5}\theta}{80}\right)$	$\left(\frac{\sqrt{5}\theta}{80},\frac{\sqrt{5}\theta}{80}\right)$
	$20\le\theta\le40$	$2x_1+x_2=\frac{\theta}{20}$	$\left(\frac{\sqrt{5}\theta}{120},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{120}\right)$	$\left(\frac{\sqrt{5}\theta}{80},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{80}\right)$	$\left(\frac{\sqrt{5}\theta}{80},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{80}\right)$
	$40<\theta<60$	$2x_1+x_2=\frac{\theta}{20}$; $x_1=x_2=0$	$\left(\sqrt{5}-\frac{\sqrt{5}\theta}{60},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{120}\right)$	$\left(\frac{3\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{40},0\right)$	$\left(\frac{3\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{40},0\right)$
Productivity inequality $(20,20,2,1)$	$0<\theta<20$	$2x_1+x_2=\frac{\theta}{20}$	$\left(\frac{\sqrt{5}\theta}{120},\frac{\sqrt{5}\theta}{60}\right)$	$\left(\frac{\sqrt{5}\theta}{120},\frac{\sqrt{5}\theta}{60}\right)$	$\left(\frac{\sqrt{5}\theta}{80},\frac{\sqrt{5}\theta}{80}\right)$
	$20\le\theta\le40$	$2x_1+x_2=\frac{\theta}{20}$	$\left(\frac{\sqrt{5}\theta}{120},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{120}\right)$	$\left(\frac{\sqrt{5}\theta}{120},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{120}\right)$	$\left(\frac{\sqrt{5}\theta}{80},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{80}\right)$
	$40<\theta<60$	$2x_1+x_2=\frac{\theta}{20}$; $x_1=x_2=0$	$\left(\sqrt{5}-\frac{\sqrt{5}\theta}{60},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{120}\right)$	$\left(\sqrt{5}-\frac{\sqrt{5}\theta}{60},\frac{\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{120}\right)$	$\left(\frac{3\sqrt{5}}{2}-\frac{\sqrt{5}\theta}{40},0\right)$
Aligned inequality $(24,12,2,1)$	$0<\theta<12$	$4x_1+x_2=\frac{\theta}{12}$	$\left(\frac{\sqrt{17}\theta}{240},\frac{\sqrt{17}\theta}{60}\right)$	$\left(\frac{\sqrt{17}\theta}{144},\frac{\sqrt{17}\theta}{72}\right)$	$\left(\frac{\sqrt{17}\theta}{96},\frac{\sqrt{17}\theta}{96}\right)$
	$12\le\theta\le48$	$4x_1+x_2=\frac{\theta}{12}$	$\left(\frac{\sqrt{17}\theta}{240},\frac{\sqrt{17}}{4}-\frac{\sqrt{17}\theta}{240}\right)$	$\left(\begin{array}{c}\min\left\{\frac{\sqrt{17}\theta}{144},\frac{\sqrt{17}}{4}\right\},\\ \max\left\{\frac{\sqrt{17}}{4}-\frac{\sqrt{17}\theta}{144},0\right\}\end{array}\right)$	$\left(\begin{array}{c}\min\left\{\frac{\sqrt{17}\theta}{96},\frac{\sqrt{17}}{4}\right\},\\ \max\left\{\frac{\sqrt{17}}{4}-\frac{\sqrt{17}\theta}{96},0\right\}\end{array}\right)$
	$48<\theta<60$	$4x_1+x_2=\frac{\theta}{12}$; $x_1=x_2=0$	$\left(\sqrt{17}-\frac{\sqrt{17}\theta}{60},\frac{\sqrt{17}}{4}-\frac{\sqrt{17}\theta}{240}\right)$	$\left(\frac{5\sqrt{17}}{4}-\frac{\sqrt{17}\theta}{48},0\right)$	$\left(\frac{5\sqrt{17}}{4}-\frac{\sqrt{17}\theta}{48},0\right)$
Misaligned inequality $(30,15,1,2)$	$0<\theta<30$	$x_1+x_2=\frac{\theta}{30}$	$\left(\frac{\sqrt{2}\theta}{60},\frac{\sqrt{2}\theta}{60}\right)$	$\left(\frac{\sqrt{2}\theta}{45},\frac{\sqrt{2}\theta}{90}\right)$	$\left(\frac{\sqrt{2}\theta}{60},\frac{\sqrt{2}\theta}{60}\right)$
	$\theta=30$	$x_1+x_2=\frac{\theta}{30}$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\left(\frac{2\sqrt{2}}{3},\frac{\sqrt{2}}{3}\right)$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
	$30<\theta<60$	$x_1+x_2=\frac{\theta}{30}$; $x_1=x_2=0$	$\left(\sqrt{2}-\frac{\sqrt{2}\theta}{60},\sqrt{2}-\frac{\sqrt{2}\theta}{60}\right)$	$\left(\begin{array}{c}\max\left\{\sqrt{2}-\frac{\sqrt{2}\theta}{90},2\sqrt{2}-\frac{\sqrt{2}\theta}{30}\right\},\\ \max\left\{\sqrt{2}-\frac{\sqrt{2}\theta}{45},0\right\}\end{array}\right)$	$\left(\sqrt{2}-\frac{\sqrt{2}\theta}{60},\sqrt{2}-\frac{\sqrt{2}\theta}{60}\right)$

Columns 4–6 in Table 1 show the (absolute) sizes of the three sets S_i^1, S_i^2, and S_i^3 for different game scenarios and ranges of θ. Overall, the difference between the sizes of the two cooperative NEs sets, $\left|S_1^k\right| - \left|S_2^k\right|$, expands as θ increases for all $k = 1, 2, 3$. In other words, higher thresholds result in an increased probability that the more 'able' player makes a greater contribution at a randomly chosen cooperative NE. However, the size relationship between $\left|S_1^k\right|$ and $\left|S_2^k\right|$ depends crucially on e_i, p_i, θ, and the contribution type.

For relative contribution, as pointed out in Theorem 2(1), in game scenarios with heterogeneity in $e_i p_i$ (i.e., the endowment inequality, productivity inequality, and aligned inequality scenarios), the size of $\left|S_1^1\right|$ is less than $\left|S_2^1\right|$ for low thresholds $0 < \theta < e_2 p_2$, whereas the opposite is true for high threshold $e_1 p_1 < \theta < G$. In addition, for intermediate thresholds $e_2 p_2 \le \theta \le e_1 p_1$, the size of $\left|S_1^1\right|$ (or $\left|S_2^1\right|$) is increasing (or decreasing) in θ. In contrast, in game scenarios without heterogeneity in $e_i p_i$ (i.e., the full equality and misaligned inequality scenarios), $\left|S_1^1\right| = \left|S_2^1\right|$ for all θ. For collective contribution, Theorem 2(3) predicts $\left|S_1^3\right| = \left|S_2^3\right|$ for low thresholds $0 < \theta < e_2 p_2$. This is indeed the case. Furthermore, we observe $\left|S_1^3\right| > \left|S_2^3\right|$ for the intermediate and high thresholds $e_2 p_2 < \theta < G$ in the three game scenarios with heterogeneity in $e_i p_i$. This observation is slightly stronger than the prediction of Theorem 2(3) (which says that $\left|S_1^3\right| \ge \left|S_2^3\right|$). Finally, the numerical analysis confirms that the results for the absolute contribution are sensitive to e_i and p_i. In particular, $|S_1^2| = |S_2^2|$ in the full equality scenario; $|S_1^2| \geq |S_2^2|$ for all θ in the endowment inequality and misaligned inequality scenarios, and $|S_1^2| < |S_2^2|$ for low thresholds and $|S_1^2| > |S_2^2|$ for high thresholds in the productivity inequality and aligned inequality scenarios.

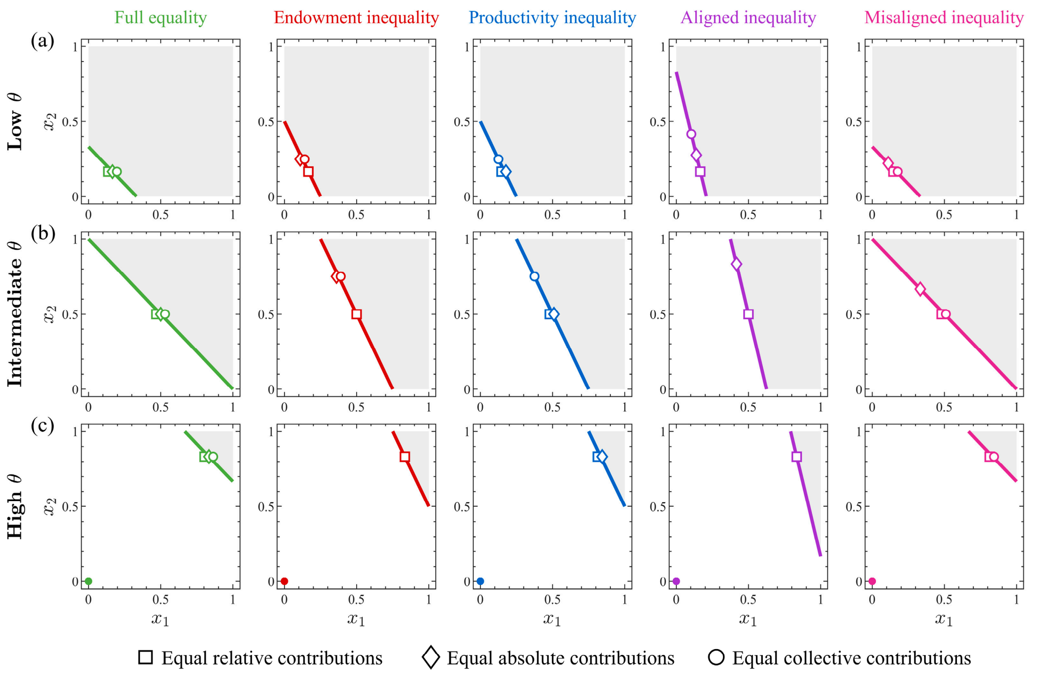

Figure 1. Nash equilibria for different game scenarios and thresholds θ. Utilizing parameters from Table 1, we set (**a**) $\theta = 10$ in the low θ panel, (**b**) $\theta = 30$ in the intermediate θ panel, and (**c**) $\theta = 50$ in the high θ panel for all five game scenarios. The colored segments denote the sets of cooperative NEs, and the colored filled points denote the defective NE. Squares, diamonds, and circles, respectively, denote the points of equal relative contributions, equal absolute contributions, and equal collective contributions in the cooperative NE set. Thus, these points are the segmentation points of sets S_1^k (below the point) and S_2^k (above the point) for $k = 1, 2, 3$. When multiple points overlap, they are adjusted horizontally to be better differentiated. The grey area indicates that the threshold is reached.

5. Concluding Remarks

In this paper, we consider an n-player asymmetric TPGG, where players have different endowments, productivities, and rewards. This game can have a defective NE and multiple cooperative NEs. We show that heterogeneity in $e_i p_i$ can promote cooperation in the sense that the existence condition of the defective NE becomes stricter. Furthermore, players with higher $e_i p_i$ are likely to contribute more at a cooperative NE in terms of collective contribution, but they do not necessarily have a higher relative contribution or absolute contribution. This result is consistent with a recent experimental study on a two-player asymmetric TPGG [11]. In the study [11], subjects either differ in their endowments or in their productivities. In most of the successful groups (i.e., the group has reached the threshold), the collective contribution of the rich individual or the more productive individual is higher, but the two group members tend to have similar relative contributions in the endowment inequality scenario, while having similar absolute contributions in the productivity inequality scenario. In sum, our study highlights the nontrivial effects of inequality on cooperation and coordination in a TPGG.

To deepen our understanding on human behaviors in TPGGs, a possible direction for future research is to empirically validate the theoretical findings for scenarios where players differ in two or more dimensions (e.g., the aligned and misaligned inequality scenarios discussed in Section 4). Another direction for a future investigation could focus on the coordination problem in an asymmetric climate game. A recent study showed that cooperation can be an equilibrium outcome if and only if the weighted average of climate risk of all countries reaches or exceeds the coefficient of emission reduction target [20]. However, the climate game can have multiple cooperative equilibria, in which countries have different responsibilities for carbon emission reduction at different equilibria. Thus, a natural question would be the following: how do countries with different endowments, climate risks, and emission reduction costs coordinate? Finally, the reward value for each player in our model is considered to be fixed. It would be interesting to investigate the TPGG model within a cooperative game framework, where rewards are assigned, according to the players' values, to each coalition. Specifically, one could calculate the stable reward allocation schemes in asymmetric TPGGs based on the Shapley value, CIS value, and nucleolus and compare the fairness and efficiency of these allocation schemes.

Author Contributions: All authors share first authorship and are sorted alphabetically. Conceptualization, Y.D. and B.Z.; methodology, all authors; validation, B.Z.; formal analysis, X.A., X.W. and B.Z.; investigation, all authors; resources, Y.D. and B.Z.; writing—original draft preparation, all authors; writing—review and editing, B.Z.; visualization, X.W.; supervision, Y.D. and B.Z.; project administration, B.Z.; funding acquisition, Y.D. and B.Z. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Theorem 1

Proof.

(i) Existence conditions of a defective Nash equilibrium

If $\boldsymbol{x} = \mathbf{0}$ is a Nash equilibrium, then no player has an incentive to contribute when others do not. Therefore, for any player i with strategy $0 < x_i \leq 1$,

$$\begin{cases} e_i \geq e_i(1 - x_i) + r_i, & \text{if } e_i p_i x_i \geq \theta, \\ e_i \geq e_i(1 - x_i), & \text{if } e_i p_i x_i < \theta. \end{cases}$$

According to the assumption of $r_i \geq e_i$, it follows that, for any player i, (1) $\theta > e_i p_i$ or (2) $\theta = e_i p_i$ and $e_i = r_i$.

Intuitively, condition (1) implies that the group cannot reach the threshold when only one player contributes. Condition (2) implies that, although the threshold can be reached if player i contributes all of his or her endowment, there is no payoff improvement for player i.

(ii) Cooperative Nash equilibria set and its existence condition

Let us suppose that the strategy profile $\boldsymbol{x}$ satisfies $\sum_{i=1}^{n} e_i p_i x_i > 0$. On the one hand, if $\sum_{i=1}^{n} e_i p_i x_i = \theta$, then it is easy to check that no player has an incentive to increase or decrease his or her contribution under the assumption of $r_i \geq e_i$. On the other hand, if $\sum_{i=1}^{n} e_i p_i x_i > \theta$ or $0 < \sum_{i=1}^{n} e_i p_i x_i < \theta$, then at least one player can obtain a higher payoff by reducing his or her contribution. Therefore, the set of cooperative NEs is $\{\boldsymbol{x} | \sum_{i=1}^{n} e_i p_i x_i = \theta\}$. In addition, the cooperative NE set is empty if and only if $\theta > G$ (i.e., the group cannot reach the threshold even if all the players contribute all of their endowments). Thus, a cooperative NE exists if and only if $0 < \theta \leq G$. □

Appendix B. Proof of Theorem 2

Proof Outline. Theorem 2 includes three parts of results, namely, results for (1) the relative contribution, (2) the absolute contribution, and (3) the collective contribution. We begin by proving part (3) and then establish part (1) based on the approach developed in part (3). Finally, part (2) can be directly obtained from part (1) and part (3).

To prove part (3), it is enough to compare the measures of two consecutive sets, $|S_i^3|$ and $|S_{i+1}^3|$. We transform the set S_i^3 to a new set W_i through an affine transformation and show $|W_i| \geq |W_{i+1}|$ for all $0 < \theta < G$ and $i = 1, \ldots, n-1$. This then implies that $|S_1^3| \geq \ldots \geq |S_n^3|$.

For part (1), when $\sum_{i=1}^{n-1} e_i p_i < \theta < G$, each set S_i^1 can be regarded as an $(n-1)$-dimensional polytope. Thus, the comparison between $|S_i^1|$ and $|S_{i+1}^1|$ can be made by directly calculating the volume of these polytopes. When $0 < \theta < e_n p_n$, we apply the approach developed in part (3) to compare $|S_i^1|$ and $|S_{i+1}^1|$. Specifically, we represent S_i^1 as $\bigcup_{j=1}^{n}\left(S_i^1 \cap S_j^3\right)$, and, therefore, $|S_i^1| = \sum_{j=1}^{n}\left|S_i^1 \cap S_j^3\right|$. It is easy to check that $S_i^1 \cap S_j^3 = \varnothing$ for $i < j$. Hence, to establish $|S_i^1| < |S_{i+1}^1|$, we only need to prove the following two lemmas (refer to Figure A1).

Lemma 1. *When $0 < \theta < e_n p_n$, $\left|S_i^1 \cap S_j^3\right| < \left|S_{i+1}^1 \cap S_j^3\right|$ for $i > j$.*

Lemma 2. *When $0 < \theta < e_n p_n$, $|S_i^1 \cap S_i^3| < |S_{i+1}^1 \cap S_{i+1}^3|$ for $i = 1, \ldots, n-1$.*

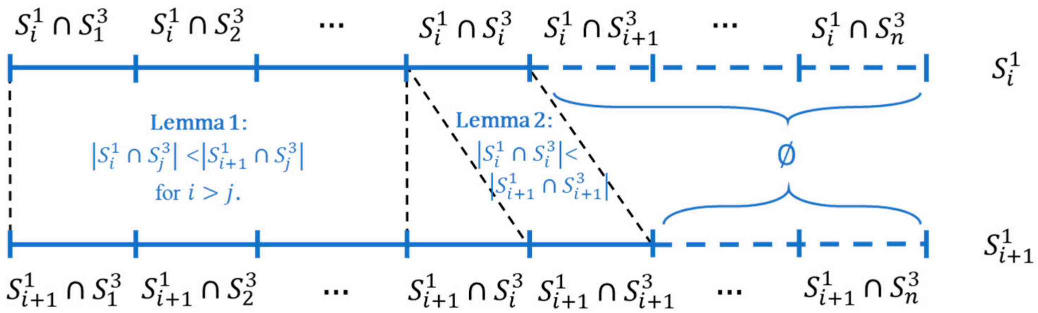

Figure A1. Comparison between $|S_i^1|$ and $\left|S_{i+1}^1\right|$ when $0 < \theta < e_n p_n$.

Proof of part (3). The set S_i^3 is defined as

$$S_i^3 = \left\{ (x_1, x_2, \ldots, x_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_k p_k x_k = \theta, \ \max\limits_k \{e_k p_k x_k\} = e_i p_i x_i, \\ x_k \in [0,1], \forall k \in \{1, 2, \ldots, n\} \end{array} \right\}.$$

Applying the following affine transformation $F: \mathbb{R}^n \to \mathbb{R}^n$ to set S_i^3 yields a new set W_i, as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} e_1 p_1 & 0 & \ldots & 0 \\ 0 & e_2 p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e_n p_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Their measures satisfy $|S_i^3| = \frac{|W_i|}{||A||}$, where $||A|| > 0$ is the Jacobian determinant. The set W_i is expressed as

$$W_i = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_k\{e_kp_kx_k\} = e_ip_ix_i, \\ 0 \le e_kp_kx_k \le e_kp_k,\ \forall k \ne i, \\ 0 \le\ e_{i+1}p_{i+1}x_{i+1} \le e_ip_ix_i \le e_ip_i \end{array} \right\}.$$

Similarly, applying the same transformation to S_{i+1}^3 yields W_{i+1}, which is expressed as

$$W_{i+1} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_k\{e_kp_kx_k\} = e_{i+1}p_{i+1}x_{i+1}, \\ 0 \le e_kp_kx_k \le e_kp_k,\ \forall k \ne i, i+1, \\ 0 \le\ e_ip_ix_i \le e_{i+1}p_{i+1}x_{i+1} \le e_{i+1}p_{i+1} \end{array} \right\}.$$

Given that $e_1p_1 > e_2p_2 > \ldots > e_np_n$, the set W_i can be partitioned into W_i' and W_i', where W_i' is expressed as

$$W_i' = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_k\{e_kp_kx_k\} = e_ip_ix_i, \\ 0 \le e_kp_kx_k \le e_kp_k,\ \forall k \ne i, i+1, \\ 0 \le\ e_{i+1}p_{i+1}x_{i+1} \le e_ip_ix_i \le e_{i+1}p_{i+1} \end{array} \right\},$$

and W_i'' is expressed as

$$W_i'' = W_i \backslash W_i'.$$

Thus, $|W_i| = |W_i'| + |W_i''|$. Due to the symmetry of $e_ip_ix_i$ and $e_{i+1}p_{i+1}x_{i+1}$, we deduce that $|W_i'| = |W_{i+1}|$. Consequently, $|W_i| \ge |W_{i+1}|$, and, therefore, $|S_i^3| \ge |S_{i+1}^3|$.

Furthermore, when $0 < \theta < e_np_n$, the set S_i^3 can be written as

$$S_i^3 = \left\{ (x_1, x_2, \ldots, x_n) \middle| \sum_{k=1}^{n} e_kp_kx_k = \theta,\ x_k \in \left[0,\ \frac{\theta}{e_kp_k}\right],\ \max_k\{e_kp_kx_k\} = e_ip_ix_i \right\}.$$

Correspondingly, the set W_i is expressed as

$$W_i = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ e_kp_kx_k \in [0,\ \theta], \\ \max\limits_k\{e_kp_kx_k\} = e_ip_ix_i \end{array} \right\}.$$

Due to the symmetry of $e_ip_ix_i$ and $e_{i+1}p_{i+1}x_{i+1}$, it is obvious that $|W_i| = |W_{i+1}|$, and, therefore, $|S_i^3| = |S_{i+1}^3|$.

Proof of part (1). The set S_i^1 is defined as

$$S_i^1 = \left\{ (x_1, x_2, \ldots, x_n) \middle| \sum_{k=1}^{n} e_kp_kx_k = \theta,\ x_k \in [0,1], \forall k \in \{1,2,\ldots,n\},\ \max_k\{x_k\} = x_i \right\}.$$

(i) When $\sum_{i=1}^{n-1} e_ip_i < \theta < G$, each S_i^1 is an $(n-1)$-dimensional convex polytope formed by n vertices. Specifically, we define the following:

$$\begin{gathered} \alpha_1 = \left(\frac{\theta - \sum_{k\ne 1}^{n} e_kp_k}{e_1p_1}, 1, 1, \ldots, 1 \right); \\ \alpha_2 = \left(1, \frac{\theta - \sum_{k\ne 2}^{n} e_kp_k}{e_2p_2}, 1, \ldots, 1 \right); \\ \vdots \\ \alpha_n = \left(1, 1, 1, \ldots, \frac{\theta - \sum_{k\ne n}^{n} e_kp_k}{e_np_n} \right); \\ \text{and } O = \left(\frac{\theta}{G}, \frac{\theta}{G}, \ldots, \frac{\theta}{G} \right). \end{gathered}$$

The vertex set of the convex polytope S_i^1 is written as $\{O, \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$. Thus, the volume of S_i^1 (in terms of the Hausdorff measure) can be obtained by calculating the magnitude of a cross product

$$\boldsymbol{\omega} = \begin{vmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \ldots & \boldsymbol{v_{i-1}} & \boldsymbol{v_i} & \boldsymbol{v_{i+1}} & \ldots & \boldsymbol{v_n} \\ a_1 & a & \ldots & a & a & a & \ldots & a \\ a & a_2 & \ldots & a & a & a & \ldots & a \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a & a & \ldots & a_{i-1} & a & a & \ldots & a \\ a & a & \ldots & a & a & a_{i+1} & \ldots & a \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a & a & \ldots & a & a & a & \ldots & a_n \end{vmatrix},$$

where $\boldsymbol{v_i}$ is the unit vector, and

$$\begin{aligned} a &= 1 - \tfrac{\theta}{G}; \\ a_1 &= \frac{\theta - \sum_{k\neq 1}^n e_k p_k}{e_1 p_1} - \tfrac{\theta}{G}; \\ a_2 &= \frac{\theta - \sum_{k\neq 2}^n e_k p_k}{e_2 p_2} - \tfrac{\theta}{G}; \\ &\vdots \\ a_n &= \frac{\theta - \sum_{k\neq n}^n e_k p_k}{e_n p_n} - \tfrac{\theta}{G}. \end{aligned}$$

The magnitude of $\boldsymbol{\omega}$, which equals the measure of S_i^1, is then given by

$$|\boldsymbol{\omega}| = \frac{\sqrt{\sum_{k=1}^n (e_k p_k)^2}\,(G-\theta)^{n-1}}{G \cdot \prod_{i=1}^n e_k p_k} e_i p_i.$$

Since $e_1 p_1 > e_2 p_2 > \ldots > e_n p_n$, it implies that $\left|S_i^1\right| > \left|S_{i+1}^1\right|$ for all $i \in \{1, 2, \ldots, n-1\}$.

(ii) When $0 < \theta < e_n p_n$, we represent S_i^1 as $\bigcup_{j=1}^n \left(S_i^1 \cap S_j^3\right)$ and focus on the set $S_i^1 \cap S_j^3$. This set can be expressed as

$$S_i^1 \cap S_j^3 = \left\{ (x_1, x_2, \ldots, x_n) \middle| \begin{array}{c} \sum_{k=1}^n e_k p_k x_k = \theta,\ x_k \in [0,1], \forall k \in \{1, 2, \ldots, n\}, \\ \max_k \{e_k p_k x_k\} = e_j p_j x_j,\ \max_k \{x_k\} = x_i \end{array} \right\}.$$

In the following, we prove the two lemmas mentioned in the proof outline.

Proof of Lemma 1. Applying the affine transformation F (this transformation is introduced in the proof of part (3)) to the set $S_i^1 \cap S_j^3$ yields a new set $V_{i,j}$,

$$V_{i,j} = \left\{ (e_1 p_1 x_1, e_2 p_2 x_2, \ldots, e_n p_n x_n) \middle| \begin{array}{c} \sum_{k=1}^n e_k p_k x_k = \theta,\ e_k p_k x_k \in [0, e_k p_k], \forall k \in \{1, 2, \ldots, n\}, \\ \max_k \{e_k p_k x_k\} = e_j p_j x_j, \max_k \{x_k\} = x_i \end{array} \right\}.$$

First, we claim that $\left|V_{i,j}\right| > 0$ for $i > j$ under the condition $0 < \theta < e_n p_n$. To demonstrate this, we prove that the interior of $V_{i,j}$ is nonempty. Given $0 < \theta < e_n p_n$ and $i > j$, we can find $\hat{\boldsymbol{q}} = (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n) \in V_{i,j}$ such that $\max_{k\neq j}\{\hat{q}_k\} < \hat{q}_j$ and $\max_{k\neq i}\left\{\frac{\hat{q}_k}{e_k p_k}\right\} < \frac{\hat{q}_i}{e_i p_i}$. For every $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$ close to $\hat{\boldsymbol{q}}$, it can be written as $\varepsilon \boldsymbol{q} + (1-\varepsilon)\hat{\boldsymbol{q}}$ for small ε. Specifically, we can choose $\boldsymbol{q}$ from the compact set

$$C = \left\{ (e_1 p_1 x_1, e_2 p_2 x_2, \ldots, e_n p_n x_n) \middle| \begin{array}{c} \sum_{k=1}^n e_k p_k x_k = \theta, e_k p_k x_k \in [0, e_k p_k], \\ e_s p_s x_s = 0 \text{ for some } s \in \operatorname{supp}(\hat{\boldsymbol{q}}) \end{array} \right\},$$

which consists of the faces which do not contain $\hat{\boldsymbol{q}}$. It is evident that $\boldsymbol{g} \in V_{i,j}$ for a sufficiently small ε. Hence, for every $\boldsymbol{q} \in C$, $\boldsymbol{g} = \varepsilon\boldsymbol{q} + (1-\varepsilon)\hat{\boldsymbol{q}} \in V_{i,j}$ for all $\varepsilon < \bar{\varepsilon}(\boldsymbol{q})$. It is easy to see that $\bar{\varepsilon}(\boldsymbol{q})$ can be chosen as continuous. Let $\bar{\varepsilon} := \min\{\bar{\varepsilon}(\boldsymbol{q}) : \boldsymbol{q} \in C\}$, which is strictly positive. Therefore, the neighborhood of $\hat{\boldsymbol{q}}$, denoted as $\{\boldsymbol{g} | \boldsymbol{g} = \varepsilon\boldsymbol{q} + (1-\varepsilon)\hat{\boldsymbol{q}}, \boldsymbol{q} \in C \text{ and } \varepsilon < \bar{\varepsilon}\}$, is contained within the set $V_{i,j}$. Consequently, $|V_{i,j}| > 0$ for $i > j$.

Next, we rewrite the sets $V_{i+1,j}$ and $V_{i,j}$ as follows:

$$V_{i+1,j} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^n e_kp_kx_k = \theta,\ e_kp_kx_k \in [0, e_kp_k], \forall k \in \{1,2,\ldots,n\}, \\ \max\limits_k\{e_kp_kx_k\} = e_jp_jx_j, \\ \max\limits_{k \neq i+1}\{e_{i+1}p_{i+1}x_k\} \leq e_{i+1}p_{i+1}x_{i+1} \leq e_jp_jx_j \end{array} \right\},$$

and

$$V_{i,j} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^n e_kp_kx_k = \theta,\ e_kp_kx_k \in [0, e_kp_k], \forall k \in \{1,2,\ldots,n\}, \\ \max\limits_k\{e_kp_kx_k\} = e_jp_jx_j, \\ \max\limits_{k \neq i}\{e_ip_ix_k\} \leq e_ip_ix_i \leq e_jp_jx_j \end{array} \right\}.$$

Partition the set $V_{i+1,j}$ into $V'_{i+1,j}$ and $V''_{i+1,j}$, where

$$V'_{i+1,j} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^n e_kp_kx_k = \theta, e_kp_kx_k \in [0, e_kp_k], \forall k \in \{1,2,\ldots,n\}, \\ \max\limits_k\{e_kp_kx_k\} = e_jp_jx_j, \\ \max\limits_{k \neq i+1}\{e_{i+1}p_{i+1}x_k\} \leq e_{i+1}p_{i+1}x_{i+1} < e_ip_i\max\left\{ \max\limits_{k \neq i,i+1}\{x_k\}, \frac{e_ip_ix_i}{e_{i+1}p_{i+1}} \right\} \end{array} \right\},$$

and

$$V''_{i+1,j} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^n e_kp_kx_k = \theta, e_kp_kx_k \in [0, e_kp_k], \forall k \in \{1,2,\ldots,n\}, \\ \max\limits_k\{e_kp_kx_k\} = e_jp_jx_j, \\ e_ip_i\max\left\{ \max\limits_{k \neq i,i+1}\{x_k\}, \frac{e_ip_ix_i}{e_{i+1}p_{i+1}} \right\} \leq e_{i+1}p_{i+1}x_{i+1} \leq e_jp_jx_j \end{array} \right\}.$$

Following this, $|V_{i+1,j}| = \left|V'_{i+1,j}\right| + \left|V''_{i+1,j}\right|$. Due to the symmetry of $e_ip_ix_i$ and $e_{i+1}p_{i+1}x_{i+1}$, we can deduce that $|V_{i,j}| = \left|V''_{i+1,j}\right|$ for $i > j$. Considering that $\max\limits_{k \neq i+1}\{e_{i+1}p_{i+1}x_k\} < e_ip_i\max\left\{ \max\limits_{k \neq i,i+1}\{x_k\}, \frac{e_ip_ix_i}{e_{i+1}p_{i+1}} \right\}$, it follows that $\left|V'_{i+1,j}\right| > 0$. Therefore, $|V_{i,j}| < |V_{i+1,j}|$, and, thus, $\left|S_i^1 \cap S_j^3\right| < \left|S_{i+1}^1 \cap S_j^3\right|$ for $i > j$.

Proof of Lemma 2. In the first step, we prove that $\left|S_m^1 \cap S_i^3\right| > \left|S_m^1 \cap S_{i+1}^3\right|$ for $m > i+1$ and $0 < \theta < e_np_n$. Let us consider the set $V_{m,i}$, which is obtained by applying the affine transformation F to set $S_m^1 \cap S_i^3$.

$$V_{m,i} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^n e_kp_kx_k = \theta,\ e_kp_kx_k \in [0, e_kp_k], \forall k \in \{1,2,\ldots,n\}, \\ \max\limits_k\{e_kp_kx_k\} = e_ip_ix_i,\ \max\limits_k\{x_k\} = x_m \end{array} \right\}.$$

When $0 < \theta < e_np_n$, from Lemma 1, we have $|V_{m,i+1}| > 0$ and $|V_{m,i}| > 0$ for $m > i+1$.

We rewrite the set $V_{m,i+1}$ and $V_{m,i}$ as

$$V_{m,i+1} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_{k}\{e_kp_kx_k\} = e_{i+1}p_{i+1}x_{i+1}, \\ \forall k \neq i, i+1,\ 0 \leq e_kp_kx_k \leq e_kp_kx_m, \\ 0 \leq e_ip_ix_i \leq e_{i+1}p_{i+1}x_{i+1} \leq e_{i+1}p_{i+1}x_m \end{array} \right\},$$

and

$$V_{m,i} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_{k}\{e_kp_kx_k\} = e_ip_ix_i, \\ \forall k \neq i,\ 0 \leq e_kp_kx_k \leq e_kp_kx_m, \\ 0 \leq e_{i+1}p_{i+1}x_{i+1} \leq e_ip_ix_i \leq e_ip_ix_m \end{array} \right\},$$

and then partition the set $V_{m,i}$ into $V'_{m,i}$ and $V''_{m,i}$, where

$$V'_{m,i} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_{k}\{e_kp_kx_k\} = e_ip_ix_i, \\ \forall k \neq i, i+1,\ 0 \leq e_kp_kx_k \leq e_kp_kx_m, \\ 0 \leq e_{i+1}p_{i+1}x_{i+1} \leq e_ip_ix_i \leq e_{i+1}p_{i+1}x_m \end{array} \right\},$$

and

$$V''_{m,i} = \left\{ (e_1p_1x_1, e_2p_2x_2, \ldots, e_np_nx_n) \middle| \begin{array}{c} \sum_{k=1}^{n} e_kp_kx_k = \theta,\ \max\limits_{k}\{e_kp_kx_k\} = e_ip_ix_i, \\ \forall k \neq i,\ 0 \leq e_kp_kx_k \leq e_kp_kx_m, \\ e_{i+1}p_{i+1}x_m < e_ip_ix_i \leq e_ip_ix_m \end{array} \right\}.$$

Hence, $|V_{m,i}| = \left|V'_{m,i}\right| + \left|V''_{m,i}\right|$. Due to the symmetry of $e_ip_ix_i$ and $e_{i+1}p_{i+1}x_{i+1}$, $|V_{m,i+1}| = \left|V'_{m,i}\right|$. Since $e_ip_ix_m > e_{i+1}p_{i+1}x_m$, $\left|V''_{m,i}\right| > 0$. Therefore, $|V_{m,i}| > |V_{m,i+1}|$, and, thus, $\left|S_m^1 \cap S_i^3\right| > \left|S_m^1 \cap S_{i+1}^3\right|$ for $m > i+1$.

In the second step, we prove $\left|S_i^1 \cap S_i^3\right| < \left|S_{i+1}^1 \cap S_{i+1}^3\right|$. From part (3), $\left|S_1^3\right| = \ldots = \left|S_n^3\right|$ when $0 < \theta < e_np_n$. We note that the following equations hold:

$$\left|S_{i+1}^3\right| = \sum_{j=1}^{n}\left|S_j^1 \cap S_{i+1}^3\right|;$$

$$\left|S_i^3\right| = \sum_{j=1}^{n}\left|S_j^1 \cap S_i^3\right|;$$

and $\left|\left(S_j^1 \cap S_i^3\right) \cap \left(S_k^1 \cap S_i^3\right)\right| = 0$ for all $j \neq k$. Given $\left|S_m^1 \cap S_i^3\right| > \left|S_m^1 \cap S_{i+1}^3\right|$ for $m > i+1$ and $S_i^1 \cap S_j^3 = \varnothing$ for $i < j$, we deduce that $\left|S_i^1 \cap S_i^3\right| + \left|S_{i+1}^1 \cap S_i^3\right| < \left|S_{i+1}^1 \cap S_{i+1}^3\right|$ (refer to Figure A2). Consequently, $\left|S_i^1 \cap S_i^3\right| < \left|S_{i+1}^1 \cap S_{i+1}^3\right|$. □

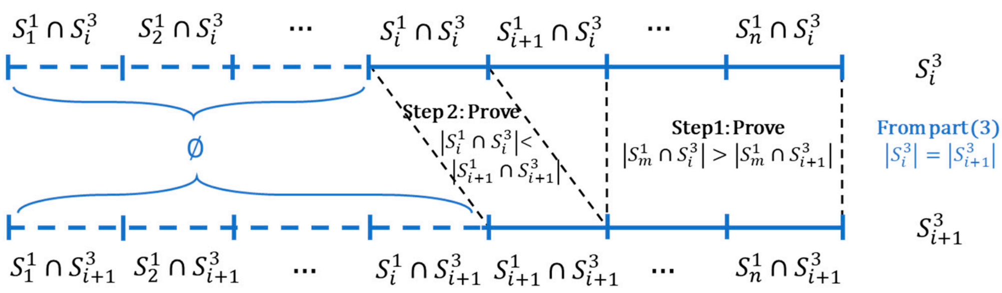

Figure A2. Comparison between $\left|S_i^1 \cap S_i^3\right|$ and $\left|S_{i+1}^1 \cap S_{i+1}^3\right|$ when $0 < \theta < e_np_n$.

Notes

[1] Dragicevic [10] theoretically studied a TPGG in the context of the option fund market and found that payoff inequality between buyers and sellers can undermine coordination efforts.

[2] Dong et al. [19] considered a climate game with two types of players, in which rich (or poor) players have higher (or lower) endowment and emission reduction cost (i.e., low productivity). Their theoretical analysis and behavioral experiment based on specific parameters showed that the effect of multiple inequalities on coordination is generally more complex. More general discussion on NE in a climate game with heterogeneous players can be found in [19].

[3] We note that at an NE, the absolute contribution of player i cannot exceed r_i even if $r_i < e_i$. Otherwise, this player can obtain a higher payoff by deviating to free-riding. Thus, this assumption does not affect the equilibrium structure of the game.

[4] An alternative scenario is one in which players choose their strategies from a finite grid $\left\{0, \frac{1}{m}, \ldots, 1\right\}$ with sufficiently large m. In this case, the cooperative NE set consists of finite number of equilibria, and $\left|S_i^k\right| > \left|S_j^k\right|$ for all $j \neq i$ ($k = 1, 2, 3$) implies that there are more equilibria in which player i contributes the most.

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