Week 4. Static games with complete information III: Nash equilibria

Exercise 1: Nash equilibrium vs dominance solvability

Prove the following statements:

(i) If a pure strategy $s_j^{(i)}$ is dominated by a pure strategy $s_k^{(i)}$ and $\sigma = (\sigma^{(1)}, \ldots, \sigma^{(n)})$ is a Nash equilibrium, then $\sigma_j^{(i)} = 0$.

(ii) If the game is dominance solvable such that the unique outcome of iterated elimination of dominated strategies is some pure strategy $s = (s^{(1)}, \ldots, s^{(n)})$, then s is a Nash equilibrium.

[Suggestion: One could use contradiction to prove the above statements. For example, for (i) assume that these was a Nash equilibrium with $\sigma_j^{(i)} > 0$, and show that this would yield some contradiction.]

Exercise 2: Best responses

Consider the stag hunt game:

$$
\begin{array}{ccc}
 & & \underline{\text{player 2}} \\
 & & \begin{array}{cc} \text{Stag} & \text{Hare} \end{array} \\
\underline{\text{player 1}} & \begin{array}{c} \text{Stag} \\ \text{Hare} \end{array} & \begin{pmatrix} (10,10) & (0,6) \\ (6,0) & (6,6) \end{pmatrix}
\end{array}
$$

Suppose player 1 uses the mixed strategy $(x, 1-x)$, where x is player 1's probability to Stag. Similarly, player 2's strategy is $(y, 1-y)$.

(i) For given x, y compute the players' payoffs $\pi^{(1)}(x, y), \pi^{(2)}(x, y)$ (see Remarks 2.6, 2.7).

(ii) For a given y compute player 1's best response $(\mathrm{BR}(y))$. In particular, show that there is some y^* such that all $x \in [0, 1]$ are a best response.

(iii) Draw the two best response correspondences $\mathrm{BR}(x)$, $\mathrm{BR}(y)$ into a $x - y$ plane. How often do they intersect? What does it mean if they intersect?

Exercise 3: Cournot Duopoly

The Cournot duopoly game is defined by:

- Players: $N = \{\text{Firm } 1, \text{Firm } 2\}$
- Actions: Amount of good produced, $x^{(i)} \in [0, \infty)$ for $i \in \{1, 2\}$
- Payoffs: $\pi^{(i)}(x^{(1)}, x^{(2)}) = [a - b(x^{(1)} + x^{(2)})]x^{(i)} - cx^{(i)}$

Show that there is a Nash equilibrium in pure strategies. For simplicity assume $a = 10, b = 1, c = 1$.

[Hint: For each $x^{(i)}$ computer $\text{BR}(x^{(-i)})$. Then solve simultaneously:

$$x^{(1)} = \text{BR}(x^{(2)})$$
$$x^{(2)} = \text{BR}(x^{(1)})$$

]

Exercise 4: Matching Pennies

Compute the Nash equilibria for the following two games, and interpret the result.

$$\begin{array}{c} \\ \text{Top} \\ \text{Bottom} \end{array} \begin{array}{c} \begin{array}{cc} \text{Left} & \text{Right} \end{array} \\ \begin{pmatrix} (0.8, 0.4) & (0.4, 0.8) \\ (0.4, 0.8) & (0.8, 0.4) \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ \text{Top} \\ \text{Bottom} \end{array} \begin{array}{c} \begin{array}{cc} \text{Left} & \text{Right} \end{array} \\ \begin{pmatrix} (3.2, 0.4) & (0.4, 0.8) \\ (0.4, 0.8) & (0.8, 0.4) \end{pmatrix} \end{array}$$

Bonus Exercise 1: Verifying NE in games with finitely many players & actions

Show that to verify whether a strategy profile $\hat{\sigma} = (\hat{\sigma}^{(1)}, \ldots, \hat{\sigma}^{(n)})$ is a Nash equilibrium, it is sufficient to check all deviations towards pure strategies.

Specifically show that $\hat{\sigma}$ is a Nash equilibrium if and only if for all players i the following two conditions hold:

(i) All actions that player i uses give the same payoff: if $\sigma_j^{(i)} > 0$ and $\sigma_k^{(i)} > 0$ then $\pi^{(i)}(s_j^{(i)}, \hat{\sigma}^{(-i)}) = \pi^{(i)}(s_k^{(i)}, \hat{\sigma}^{(-i)})$.

(ii) Actions that are not played are not profitable: if $\sigma_j^{(i)} = 0$ then $\pi^{(i)}(s_j^{(i)}, \hat{\sigma}^{(-i)}) \leq \pi^{(i)}(\hat{\sigma}^{(i)}, \hat{\sigma}^{(-i)})$.

[Hint: One way to prove the above is once again by contradiction.]

Bonus Exercise 2: Finding games with a non-generic number of equilibria

Find an example of a symmetric 2 player game, with 2 actions per player, with:

- Exactly 2 Nash equilibria
- infinitely many Nash equilibria

[Note: These should include all Nash equilibria. Not just pure Nash equilibria.]