Week 6. Sequential games with complete information II: Subgame perfection and repeated games

Exercise 1: Battle of the sexes with an outside option

The battle of sexes is a game where two people would prefer to do something together, but each person likes a different activity best. The payoff matrix is

$$\begin{array}{cc} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} (3,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix} \end{array}$$

Now suppose that before playing this game, player 1 can choose whether to play this game or to exit. If player 1 exits, both players obtain a payoff of 2.

Show that the battle of sexes with an outside option has two pure subgame perfect equilibria:

(i) Player 1 plays, and both players choose activity a_1.

(ii) Player 1 exits, because if they were to play they would both choose activity a_2.

Bonus question: Can you give a compelling argument why player 1 may be able to undermine the second equilibrium?

Exercise 2: Two rounds of rock-paper-scissors

Consider the game rock, paper, scissors with the following matrix

$$\begin{array}{cc} & \begin{array}{ccc} \text{rock} & \text{paper} & \text{scissors} \end{array} \\ \begin{array}{c} \text{rock} \\ \text{paper} \\ \text{scissors} \end{array} & \begin{pmatrix} (0,0) & (-1,1) & (1,-1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix} \end{array}$$

(i) If this game is played once, show that the strategy profile where both players use the mixed strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the unique Nash equilibrium (You do not need to show uniqueness, only that it is a Nash equilibrium).

(ii) What is the subgame perfect equilibrium if the game is played twice?

Interpret the result.

Bonus 1: Cooperation in a finitely repeated game

Consider the game with the payoff matrix

$$\begin{array}{cc} & \begin{array}{ccc} C & D_1 & D_2 \end{array} \\ \begin{array}{c} C \\ D_1 \\ D_2 \end{array} & \begin{pmatrix} (3,3) & (0,4) & (-12,0) \\ (4,0) & (1,1) & (-10,0) \\ (0,-12) & (0,-10) & (-5,-5) \end{pmatrix} \end{array}$$

Show that:

(i) If the game is only played once, there is <u>no</u> Nash equilibrium in which C is played with a positive probability.

(ii) If the game is played twice, there is a subgame perfect equilibrium in which C is played in the first round.

[Hint: Consider the strategy: Play C in the first round. If both players played C in the first round play D_1 in the second round, otherwise play D_2.]

Bonus Exercise 2: Repeated prisoner's dilemma

Consider the infinitely repeated prisoner's dilemma with payoffs,

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,1) & (1,1) \end{pmatrix} \end{array}$$

Prove that there are sequences of actions such that players' average payoff,

$$\frac{1}{T+1}\sum_{t=0}^{T} u^{(i)}(a_t)$$

does not converge as $T \to \infty$.

[Hint: Consider the case that both players first play C for one round. Then they play D for 2 rounds. Then they play C for four rounds. Then they play D for 8 rounds, etc]