Week 7. Sequential games with complete information III: Repeated games

Exercise 1: Win-Stay Lose-Shift

The strategy Win-Stay Lose-Shift (WSLS) cooperates in the first round. In all subsequent rounds, it cooperates if either both players cooperated in the previous round, or if no one did. Otherwise it defects.

Show that for the infinitely repeated prisoner's dilemma with stage game payoffs

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

the strategy profile (WSLS, WSLS) is a subgame perfect equilibrium if $\delta \geq \frac{1}{2}$.

[Hint: Similarly to the examples covered in class, to prove that the strategy profile (WSLS, WSLS) is a subgame perfect equilibrium we need to check different cases. For case one consider a history h_t according to which either both players cooperated in the previous round, or both players defected. For case two consider that one player defected.]

Exercise 2: Mini-Max I

Consider the matching pennies games

$$\begin{array}{cc} & \begin{array}{cc} \text{Left} & \text{Right} \end{array} \\ \begin{array}{c} \text{Up} \\ \text{Down} \end{array} & \begin{pmatrix} (0.8,0.4) & (0.4,0.8) \\ (0.4,0.8) & (0.8,0.4) \end{pmatrix} \end{array}$$

Show that in the definition of minimax, it is important to allow for mixed strategies of the opponent.

Specifically show that:

$$\min_{s^{(2)}} \max_{s^{(1)}} u^{(1)}(s^{(1)}, s^{(2)}) = 0.8, \text{ but}$$
$$\min_{\sigma^{(2)}} \max_{s^{(1)}} u^{(1)}(s^{(1)}, \sigma^{(2)}) = 0.6.$$

Bonus 1: Mini-Max II

Show that the minimax payoff of a player can be lower than what this player could get in a Nash equilibrium. Specifically, consider the game

$$\begin{array}{r} \\ \text{Up} \\ \text{Medium} \\ \text{Down} \end{array} \begin{array}{c} \begin{array}{cc} \text{Left} & \text{Right} \end{array} \\ \begin{pmatrix} (-2,2) & (1,-2) \\ (1,-2) & (-2,2) \\ (0,1) & (0,1) \end{pmatrix} \end{array}$$

- **Show that** the Nash equilibria of this game are of the form

$$\sigma^{(1)} = (0,0,1)$$
$$\sigma^{(2)} = (q, 1-q) \text{ with } q \in \left[\frac{1}{3}, \frac{2}{3}\right],$$

  and the resulting payoffs are $\hat{u}^{(1)} = 0, \hat{u}^{(2)} = 1$.

- **Show that** player 2's minimax payoff $\underline{u}^{(2)} = \min_{\sigma^{(1)}} \max_{s^{(2)}} u^{(2)}(\sigma^{(1)}, s^{(2)}) = 0 < \hat{u}^{(2)}$.

Now that you have shown that the minimax payoff of a player can be lower than their Nash equilibrium payoff, **conclude that in repeated games with a sufficient large δ, players may be worse off in equilibrium than in the one shot game.**

Bonus Exercise 2: Folk Theorem

Consider the battle of the sexes

$$\begin{array}{r} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{pmatrix} (3,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix} \end{array}$$

What is the set of feasible and individually rational payoffs?

Bonus: Construct a strategy $\hat{\sigma}$ for the repeated battle of sexes that for a sufficiently large δ satisfies the following 2 conditions.

(i) When both player adopt the strategy, they obtain a payoff of approximately $\pi^{(1)} = \pi^{(2)} = 2$.

(ii) $(\hat{\sigma}, \hat{\sigma})$ is a subgame perfect equilibrium. You do not need to show this rigorously, but give a convincing argument.