GAME THEORY

Administrative stuff

1) Exercises
2) Either "Oral exam" or "final project"

Summary

1) Game theory is about strategic decision-making
   (Poker ✓, Roulette ✗)
2) Elements: Players, Actions, order of moves, payoffs, information
3) Actions: What each player can do
   Strategies: Plan which action to pick given the information the player currently has

§2 Static games with complete information (SGCI)

Definition 2.1 (SGCI, "one-shot games", "normal-form games")

SGCI are those games for which

(1) players have all payoff-relevant information

(2) Players move simultaneously.

(or in ignorance of what other players did.)

Elements of SGCI:

1) Players $\mathcal{N} = \{1, \dots, n\}$

2) Action of player i is an element of

$$A^{(i)} = \{a_1^{(i)}, \dots, a_{j_i}^{(i)}\}$$

$$\mathcal{A} = A^{(1)} \times \dots \times A^{(n)}$$

$a \in \mathcal{A}$ $\quad a = (a_1, \dots, a_n)$

3) Payoffs $\pi : \mathcal{A} \to \mathbb{R}^n$

Remark 2.2 (Two-player games with finitely many actions)

$$\mathcal{N} = \{1, 2\}$$

$$A^{(1)} = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$$

$$A^2 = \{\beta_1, \beta_2, \dots, \beta_k\}$$

$$\pi : A^{(1)} \times A^{(2)} \to \mathbb{R}^2$$

$\rightsquigarrow$ Usually written as a payoff matrix

	β_1	β_2	$\dots$	β_k
α_1	$\pi^{(1)}_{11}, \pi^{(2)}_{11}$			
α_2				
$\vdots$				
α_m	$\boxed{}$ $\pi^{(1)}_{m1}, \pi^{(2)}_{m1}$			$\pi^{(1)}_{mk}, \pi^{(2)}_{mk}$

Similar way how to represent 3-player games

$\rightsquigarrow$ Exercises

Example 2.3 (Prisoner's dilemma)

Two players, they can either confess or remain silent

Payoffs $\hat{=}$ saved prison time

	silent	confess
silent	3,3	0,4
confess	4,0	1,1

	S	C
S	3	0
C	4	1

This game has the following properties

1) $A^{(1)} = A^{(2)}$

2) $\pi^{(1)}(a, a') = \pi^{(2)}(a', a)$

2-player games with this property are called "symmetric"

n-player games? $\rightsquigarrow$ Exercises

Definition 2.4 (Strategies for SGCI)

1) Let $\Gamma = (\mathcal{N}, \mathcal{A}, \pi)$ be a SGCI.

   Let $A^{(i)} = \{a_1^{(i)}, \dots, a_m^{(i)}\}$

   Then a strategy for player i is a probability distribution over the set of actions

   $$\sigma^{(i)} = (\sigma_1^{(i)}, \sigma_2^{(i)}, \dots, \sigma_m^{(i)}) \qquad \sigma_j \geq 0 \quad \sum_j \sigma_j = 1$$

   Set of all strategies $\Sigma^{(i)}$

   Set of "strategy profiles": $\Sigma = \Sigma^{(1)} \times \Sigma^{(2)} \times \dots \times \Sigma$

2) A strategy is called pure

   if there is a j such that $\sigma_j^{(i)} = 1$, $\sigma_k^{(i)} = 0$ for all $k \neq j$.

   If I would like to highlight that a strategy is pure then I write s $\quad [s = (0, \dots, 0, 1, 0, \dots, 0)]$

   Set of all pure strategies of player i is $S^{(i)}$

Example 2.5 (Prisoner's dilemma)

Probability to remain silent

Strategy for player 1: $(1/2, 1/2)$

Probability to confess

Pure strategies $(1,0)$

$(0,1)$

Remark 2.6 (Payoffs for mixed strategies)

(*) If all players use pure strategies the resulting payoff is $\pi(a^{(1)}, a^{(2)}, \dots, a^{(n)})$

(*) I want $\pi: \Sigma \to \mathbb{R}^n$

(*) Idea: Each player randomizes independently
Strategies are $\sigma^{(i)} = (\sigma_j^{(i)})$ $\quad \sigma := (\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)}) \in \Sigma$

$$\pi^{(i)}(\sigma) := \sum_{j_1, \dots, j_n} \sigma_{j_1}^{(1)} \cdot \sigma_{j_2}^{(2)} \cdot \dots \cdot \sigma_{j_n}^{(n)} \; \pi^{(i)}(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \dots, a_{j_n}^{(n)})$$

Example 2.7 (Prisoner's dilemma)

$A^{(i)} = \{\text{Silent}, \text{Confess}\}$

	S	C
S	3	0
C	4	1

$\sigma^{(1)} = (0.6, 0.4)$

$\sigma^{(2)} = (0.3, 0.7)$

Possible outcomes	Probability	Payoff of 1
(Silent, Silent)	$0.6 \cdot 0.3$	3
(Silent, Confess)	$0.6 \cdot 0.7$	0
(Confess, Silent)	$0.4 \cdot 0.3$	4
(Confess, Confess)	$0.4 \cdot 0.7$	1

$$
\begin{aligned}
\Pi^{(1)}(\sigma^{(1)}, \sigma^{(2)}) = \ & 0.6 \cdot 0.3 \cdot 3 \\
& + 0.6 \cdot 0.7 \cdot 0 \\
& + 0.4 \cdot 0.3 \cdot 4 \\
& + 0.4 \cdot 0.7 \cdot 1
\end{aligned}
$$

$$
\begin{array}{cc|cc}
 & & 0.3 & 0.7 \\
 & & S & C \\
\hline
0.6 & S & 3^{\,0.3 \cdot 0.6} & 0^{\,0.7 \cdot 0.6} \\
0.4 & C & 4 & 1
\end{array}
$$

$$\underbrace{}_{P}$$

$$\sigma^{(1)} \cdot P \sigma^{(2)T}$$

Notation 2.8

If I want to speak of player i's strategy specifically I sometimes write $(\sigma^{(i)}, \sigma^{(-i)}) = (\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(n)})$

Similarly $\Pi^{(i)}(\sigma^{(1)}, \ldots, \sigma^{(n)}) = \Pi^{(i)}(\sigma^{(i)}, \sigma^{(-i)})$

Remark 2.9 (Solution concepts)

So far: Introduce machinery to describe a game.

Question: What does it mean to solve a game?

[What would rational players do?]

Solution concepts

(1) Elimination of dominated strategies

(2) Nash equilibria

§ 2.1 Iterated elimination of dominated strategies

Example 2.10 (Prisoner's dilemma)

	Silent	Confess
Silent	3,3	0,4
Confess	4,0	1,1

For rational players, (Confess, Confess) seems to be the only logical outcome.

Definition 2.11 (Dominated strategies)

(1) A ~~pure~~ strategy $s^{(i)}$ is strictly dominated

Suggestion $\Pi(s^{(i)}, s^{(-i)}) < \Pi(\tilde{s}^{(i)}, s^{(-i)})$

to [illegible]

is not some restriction

where does this live

---

if there is some $\sigma^{(i)} \in \Sigma^i$ such that for all $s^{(-i)} \in S^{(-i)}$:

$$\pi^{(i)}(s^{(i)}, s^{(-i)}) < \pi(\sigma^{(i)}, s^{(-i)})$$

(2) A pure strategy $s^{(i)}$ is weakly dominated if

$$\pi^{(i)}(s^{(i)}, s^{(-i)}) \leq \pi^{(i)}(\sigma^{(i)}, s^{(-i)}) \quad \forall s^{(-i)}$$

$$\exists \tilde{s}^{(-i)} : \pi^{(i)}(s^{(i)}, \tilde{s}^{(-i)}) < \pi^{(i)}(\sigma^{(i)}, \tilde{s}^{(-i)}).$$

$\sigma^{(i)} := s^{(i)}$

Remark 2.12 (On dominated strategies)

1) In prisoner's dilemma, "Silent" is strictly dominated.

2) Why do we explicitly allow for domination by mixed strategies.

These are games where no strategy is dominated if you only allow domination by pure strategies, but still domination by mixed strategies is possible.

	L	R
T	3,0	0,0
M	1,1	1,1
D	0,0	3,0

Strategy M is dominated

⇝ Exercises

3) There is no added advantage of replacing $s^{(-i)}$ by $\sigma^{(-i)}$

If statement is true for all $s^{(-i)}$ it is also true for all $\sigma^{(-i)}$.

Example 2.13 (A prisoner's dilemma with remorse)

	Silent (2)	Confess
~~Silent~~	~~3,3~~	~~0,0~~ (1)
Confess	4,0	1,1