GAME THEORY #4

Reminder

(*) Static games with complete information (SGCI) - normal form games, one-shot games

$\Gamma = (\mathcal{N}, \mathcal{A}, \pi)$

$\mathcal{N}$ ... set of players $\mathcal{N} = \{1, \dots, n\}$

$\mathcal{A}$ ... set of action profiles $\mathcal{A} = A^{(1)} \times A^{(2)} \times \dots \times A^{(n)}$

π ... payoff function $\pi: \mathcal{A} \to \mathbb{R}^n$

(*) Solution concept: $\Phi: \Gamma \to \Sigma$

(*) Iterated elimination of dominated strategies

Advantage: Solution is unique

Disadvantage: Does not solve all games

(*) Stag-Hunt game

	Stag	Hare
Stag	$\boxed{10}$,10	0,6
Hare	6,0	$\boxed{6}$,6

(Stag, Hare) this is an unlikely solution

Definition 2.20 (Nash equilibrium)

(1) Consider a game $\Gamma = (\mathcal{N}, \mathcal{A}, \pi)$. Then a strategy profile $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$ is a NE if for all players i

$$\pi^{(i)}(\hat{\sigma}^{(i)}, \hat{\sigma}^{(-i)}) \geq \pi^{(i)}(\sigma^{(i)}, \hat{\sigma}^{(-i)}) \quad \forall \sigma^{(i)} \in \Sigma^{(i)}$$

(2) The NE is called strict if the inequality is strict for all $\sigma^{(i)} \neq \hat{\sigma}^{(i)}$.

(3) The NE is called pure if all players use pure strategies. Otherwise the NE is called mixed.

Remark 2.21 (An equivalent definition)

(1) Consider an arbitrary strategy profile $\tilde{\sigma}$. Then $\hat{\sigma}^{(i)}$ is called a best response if

$$\pi^{(i)}(\hat{\sigma}^{(i)}, \tilde{\sigma}^{(-i)}) = \max_{\sigma^{(i)}} \pi^{(i)}(\sigma^{(i)}, \tilde{\sigma}^{(-i)})$$

I write $\hat{\sigma}^{(i)} \in BR^{(i)}(\tilde{\sigma})$

$\hat{\sigma}$ is NE $\Leftrightarrow$ $\forall i$ $\hat{\sigma}^{(i)} \in BR^{(i)}(\hat{\sigma})$

(2) Best responses do not need to be unique
⇝ Exercise

If $\hat{\sigma}$ is a NE such that $BR^{(i)}(\hat{\sigma})$ is unique $\forall i$
$\Leftrightarrow$ strict NE

Remark 2.22 (NE vs. iterated elimination of dominated strategies)

(1) $\tilde{\sigma} = (\tilde{\sigma}^{(1)}, \dots, \tilde{\sigma}^{(n)})$

If you would like to rule out $\tilde{\sigma}$ using the concept of elimination of dominated strategies, you need to show there is one player i and one strategy $\hat{\sigma}^{(i)}$ such that

$$\pi^{(i)}(\hat{\sigma}^{(i)}, \sigma^{(-i)}) > \pi^{(i)}(\tilde{\sigma}^{(i)}, \sigma^{(-i)}) \quad \forall \sigma^{(-i)}$$

If I want to rule out $\tilde{\sigma}^{(i)}$ with NE

$$\pi^{(i)}(\hat{\sigma}^{(i)}, \tilde{\sigma}^{(-i)}) > \pi^{(i)}(\tilde{\sigma}^{(i)}, \tilde{\sigma}^{(-i)}) \text{ for particular } \tilde{\sigma}$$

(2) If some strategy $s_j^{(i)}$ is dominated and $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$ is a Nash equilibrium then $\hat{\sigma}_j^{(i)} = 0$.

(3) If a game is dominance solvable and the solution is $s = (s^{(1)}, \dots, s^{(n)})$ then s is a NE.

} Exercise

Example (Prisoner's dilemma)

	Silent	Confess
Silent	3,3	0,4
Confess	4,0	1,1

Silent is a dominated strategy

NE: $((0,1), (0,1))$

Example 2.23

(1) Stag hunt

	Stag	Hare
Stag	$\boxed{10}$,$\boxed{10}$	0,6
Hare	6,0	$\boxed{6}$,$\boxed{6}$

$BR^{(1)}$, $BR^{(2)}$

Good news: We can solve this game although this game is not dominance solvable

Bad news: More than one solution

(2) Penalty kicks in soccer

$\mathcal{N} = \{\text{Striker}, \text{Goalkeeper}\}$

$A^{(1)} = \{\text{Shoot left}, \text{Shoot right}\}$

$A^{(2)} = \{\text{Jump left}, \text{jump right}\}$

For payoffs I make the following 2 assumptions:

1) If shot is on target, goalkeeper is correct, goalkeeper prevents goal with 60%

2) Strikers have preferred side

Left ⇝ 90% on target

Right ⇝ 80% on target

		Goalkeeper Left	Goalkeeper Right
Striker	Left	0.36, $\boxed{0.64}$	$\boxed{0.9}$, 0.1
	Right	$\boxed{0.8}$, 0.2	0.32, $\boxed{0.68}$

$BR^{(1)}$, $BR^{(2)}$

There is no obvious NE ↯

Theorem 2.24 (Nash 1950)

Every game $\Gamma = (\mathcal{N}, \mathcal{A}, \pi)$ with finitely many players and finitely many actions has at least one equilibrium. (possibly in mixed strategies)

Remark 2.25 (Proof)

Later.

Remark 2.26 (How to find NE)

(1) At first, even verifying that $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$ is a NE is not trivial.

For each player you need to check uncountably many $\sigma^{(i)}$.

(2) A useful tool to simplify this verification task is:

Claim: A strategy profile $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$ is NE if and only if the following two conditions hold:

(i) "All actions that are played give the same payoff"

If $\hat{\sigma}_k^{(i)} > 0$ and $\hat{\sigma}_j^{(i)} > 0$ then $\pi^{(i)}(s_k, \hat{\sigma}^{(-i)}) = \pi^{(i)}(s_j, \hat{\sigma}^{(-i)})$

(ii) "Actions that are not played are not more profitable"

If $\hat{\sigma}_k^{(i)} = 0$ then $\pi^{(i)}(s_k, \hat{\sigma}^{(-i)}) \leq \pi^{(i)}(\hat{\sigma}^{(i)}, \hat{\sigma}^{(-i)})$

} Exercise

Two advantages

(a) Only need to check pure strategies

(b) Condition (i) helps you to find the right probabilities

Example 2.27 (Penalty kicks)

		Goalkeeper Left	Goalkeeper Right
Striker	Left	0.36, 0.64	0.9, 0.1
	Right	0.8, 0.2	0.32, 0.68

We already know: No pure NE

$\sigma^{(1)} = (x, 1-x) \quad 0 < x < 1$

$\sigma^{(2)} = (y, 1-y) \quad 0 < y < 1$

Striker

$\pi^{(1)}(\text{Left}, \sigma^{(2)}) = 0.36y + 0.9(1-y)$

$\pi^{(1)}(\text{Right}, \sigma^{(2)}) = 0.8y + 0.32(1-y)$

$y^* = 29/51 \approx 56.8\%$

Goalkeeper

$\pi^{(2)}(\text{Left}, \sigma^{(1)}) = 0.64x + 0.2(1-x)$

$\pi^{(2)}(\text{Right}, \sigma^{(1)}) = 0.1x + 0.68(1-x)$

$x^* = 8/17 \approx 47.1\%$

[Expected payoff in equilibrium for striker 59.29%]

Remark 2.28 (How to find all NE)

(1)

	a	b	c
A	$\boxed{10}$,$\boxed{10}$	0,0	0,0
B	0,0	4,$\boxed{5}$	$\boxed{5}$,4
C	0,0	$\boxed{5}$,4	4,$\boxed{5}$
~~D~~	~~0,$\boxed{0}$~~	~~0,$\boxed{0}$~~	~~0,$\boxed{0}$~~

(i) Look at (pure) best responses
⇒ One pure NE

(ii) Eliminate dominated strategies

(iii) Go through all possible mixtures

$\sigma^{(1)} = (0, x, 1-x, 0)$ ⇝ $\pi^{(1)}(B, \sigma^{(2)}) = \pi^{(1)}(C, \sigma^{(2)})$

$\sigma^{(2)} = (y_1, y_2, 1-y_1-y_2)$

$\sigma^{(1)} = (x, 0, 1-x, 0)$

$\sigma^{(1)} = (x_1, x_2, 1-x_1-x_2, 0)$

(2) Algorithmic game theory:

Best known algorithm to find NE in 2-player games has a runtime that is exponential in the number of actions.

Remark 2.29 (Fun facts about NE)

(i) For games with finitely many players and actions, the number of equilibria is almost always finite & odd.

Exceptions are possible ⇝ Exercises

(ii) One can extend Nash's existence result to games with continuous action sets

In particular there is a Nash equilibrium in the Cournot duopoly game ⇝ Exercises

(iii) What to do if there are multiple equilibria ("Equilibrium selection")

	Stag	Hare
Stag	10,10	0,6
Hare	6,0	6,6

Payoff-dominant: 10,10

Risk-dominant: 6,6

10,10	0,4
4,0	4,4 → Max

Risk-dom.: Stag: 10/2, Hare 4

Harsanyi & Selten

(*) Payoff dominance: If one equilibrium gives a higher payoff to all players, they should pick it

(*) Risk dominance: Choose the action that gives you a higher payoff if co-player chooses 50:50

(*) Schelling: Sometimes players use outside information to coordinate.

"Focal point"

Remark 2.30 (On the status of NE concept)

(1) Today, NE is the standard solution for normal form games

Does this mean they think (Stag, Hare) is impossible?

If the 2 players engaged in pre-play communication they would never agree on (Stag, Hare)

(2) Some support in favor of NE from evolutionary game theory

"Folk theorem of EGT"

(1) Any NE is a fixed point of replicator dynamics.

(2) Any stable fixed point of Replicator dynamics is Nash.

(3) Behavioral game theory

(*) Real soccer play is consistent with NE for penalty kicks

Same goes for tennis services

(*) Goeree & Holt (2001)

"Matching Pennies"

	Left	Right
Up	0.80,0.40	0.40,0.80
Down	0.40,0.80	0.80,0.40

	Left	Right
Up	3.20,0.40	0.40,0.80
Down	0.40,0.80	0.80,0.40

For both games NE predicts row player should choose 50:50
⇝ Exercise

Humans: 1st game row-player ≈ 50:50

2nd game row-player ≈ 96:4