GAME THEORY #7

Reminder

(*) Multi-stage games with observed actions

(-) Players $N = \{1,\dots,n\}$ play for $0,1,\dots,T$ stages

(-) In each stage they move simultaneously and they know what happened before.

History at (the beginning of) stage t: $h_t = (a_0, a_1, \dots, a_{t-1})$

H_t … set of all possible histories

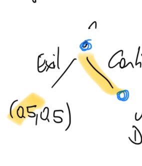
(-) Actions depend on previous play $A^{(i)}(h_t)$

(-) Payoffs: Map that depends on the entire history

$\pi: H_{T+1} \to \mathbb{R}^n$

$\pi(a_0, a_1, \dots, a_T) \mapsto (\pi^{(1)}, \dots, \pi^{(n)})$

(*) Example: Matching pennies with outside option



	L	R
U	0.8, 0.4	0.4, 0.8
D	0.4, 0.8	0.8, 0.4

⇝ Nash equilibrium (0.6, 0.6)

⇝ Started with the last subgame

(*) Subgame perfect equilibrium:

In every possible subgame you need to play a Nash equilibrium.

(*) Repeated games:

In every stage, players interact in the same normal-form game $\Gamma = (N, \mathcal{A}, u)$ ("stage game")

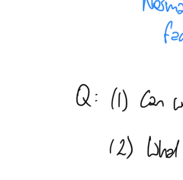
(-) Finitely repeated game $(T < \infty)$

$$\pi^{(i)} = \frac{1}{T+1} \sum_{t=0}^{T} u^{(i)}(a_t)$$

	C	D_1	D_2
C	3,3	0,4	-12,0
D_1	4,0	1,1	-10,0
D_2	0,-12	0,-10	-5,-5

⇝ Game played once ⇝ No C

⇝ Game played twice ⇝ First round C is possible



Question: If there is only one Nash equilibrium, is there anything exciting that can happen?

	C	D
C	3,3	0,4
D	4,0	1,1

⇝ No if T is finite and known.

(-) Infinitely repeated game $(T \to \infty)$

$$\pi^{(i)} = (1-\delta) \sum_{t=0}^{\infty} \delta^t u^{(i)}(a_t)$$

$(1-\delta)$ … Normalization factor

δ … discount factor < 1, continuation probability

Q: (1) Can we get cooperation in the repeated prisoner's dilemma?

(2) What can we say about all equilibria?

Remark 3.18 (How to prove subgame perfection in ∞-repeated games)

Suppose we have a strategy profile $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$

We suspect this to be a SPE. How to prove it? 2 Problems:

(1) Infinitely many subgames $G(h_t)$

(2) Even if I fix h_t, there are infinitely many ways how I could choose some deviation strategy.

⇝ Compute continuation payoffs

$$\pi^{(i)} = (1-\delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} u^{(i)}(a_\tau)$$

Fortunately this second problem can be simplified.

One-step deviation principle:

Strategy $\hat{\sigma}$ is a subgame perfect equilibrium iff there is no profitable one-step deviation

(I deviate for one round, and then play according to $\hat{\sigma}^{(i)}$)

Examples 3.19 (SPE for the repeated prisoner's dilemma)

	C	D
C	3,3	0,4
D	4,0	1,1

1) Grim/Trigger

$$\sigma^{(i)}(h_t) = \begin{cases} C & \text{if } t=0 \text{ or if } t>0 \text{ and } a_\tau^{(j)} = C \ \forall j\ \forall \tau \\ D & \text{otherwise} \end{cases}$$

(Grim, Grim)

Claim: Grim/Trigger is a SPE if $\delta \geq 1/3$.

Proof: Case 1: History is such that Grim would cooperate

	t	t+1	t+2
Grim	C	C	C
Grim	C	C	C
Payoffs	(3,3)	(3,3)	(3,3)

$\pi^{(1)} = (1-\delta)[3 + 3\delta + 3\delta^2 + \dots$

$= (1-\delta)\frac{3}{1-\delta} = 3$

One-step deviation

	t	t+1	t+2
Player 1	D	D	D
Grim	C	D	D
Payoffs	(4,0)	(1,1)	(1,1)

$\pi^{(1)} = (1-\delta)[4 + \delta \cdot 1 + \delta^2 \cdot 1 + \delta^3 \cdot 1 + \dots]$

$= (1-\delta)[4 + \frac{\delta}{1-\delta}] = (1-\delta)4 + \delta$

$= 4 - 3\delta$

$4 - 3\delta \leq 3 \iff 3\delta \geq 1$

$\delta \geq 1/3$

Case 2: History is such that Grim would defect.

	t	t+1	t+2
Grim	D	D	D
Grim	D	D	D

⇝ $\pi^{(1)} = 1$

One-step deviation:

	t	t+1	t+2	
Player 1	C	D	D	
Grim	D	D	D	
	(0,4)	(1,1)	(1,1)	…

$\pi^{(1)} = (1-\delta)[0 + \delta \cdot 1 + \delta^2 \cdot 1 + \delta^3 \cdot 1]$

$= (1-\delta) \cdot \frac{\delta}{1-\delta} = \delta \leq 1$ by assumption

2) Tit-for-Tat (TFT)

$$\hat{\sigma}^{(i)}(h_t) = \begin{cases} C & \text{if } t>0 \text{ and } a_{t-1}^{(-i)} = C \text{ or if } t=0 \\ D & \text{otherwise} \end{cases}$$

(TFT, TFT)

Claim: TFT is in general not SPE

Proof: Case 1: (C, C) in previous round
[history is such that both players would cooperate]

	t	t+1	t+2
TFT	C	C	C
TFT	C	C	C

⇝ $\pi^{(1)} = 3$

One-step deviation

	t	t+1	t+2	t+3
Player 1	D	C	D	C
TFT	C	D	C	D
	(4,0)	(0,4)	(4,0)	(0,4)

$\pi^{(1)} = (1-\delta) \cdot [4 + \delta 0 + \delta^2 4 + \delta^3 0 + \delta^4 4 \dots]$

$= (1-\delta)[4 + \underbrace{\delta^2 4 + \delta^4 4 + \delta^6 4 \dots}]$, with $4 + \delta^2 4 + \dots = \frac{4}{1-\delta^2}$

$= (1-\delta) \cdot \frac{4}{(1-\delta)(1+\delta)} = \frac{4}{1+\delta}$

$3 \geq \frac{4}{1+\delta} \iff 3 + 3\delta \geq 4$

$\iff 3\delta \geq 1$

$\delta \geq 1/3$

Case 2: Suppose $a_{t-1} = (C, D)$

	t	t+1	t+2
TFT	$\boxed{D}$	C	D
TFT	$\boxed{C}$	D	C
	(4,0)	(0,4)	(4,0)

$\pi^{(1)} = \frac{4}{1+\delta}$

One-step deviation

	t	t+1	t+2
Player 1	C	C	C
TFT	C	C	C
	(3,3)	(3,3)	(3,3)

$\pi^{(1)} = 3$

$\delta \leq 1/3$

3) Win Stay Lose Shift ⇝ Exercise

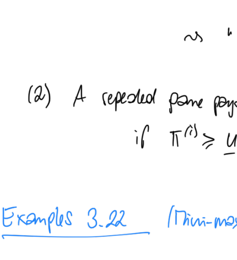
$$\sigma^{(i)}(h_t) = \begin{cases} C & \text{if } t=0 \text{ or } t>0 \text{ and } a_{t-1} \in \{(C,C),(D,D)\} \\ D & \end{cases}$$

Claim: WSLS is a SPE if $\delta > 1/2$

Remark 3.20 (Which payoffs are possible in equilibrium?)

	C	D
C	3,3	0,4
D	4,0	1,1

Let's represent this game in payoff space:



2 Necessary restrictions on payoffs that can be sustained in equilibrium

(1) "Payoffs are feasible":
$(\pi^{(1)}, \pi^{(2)})$ needs to be in the convex hull of the stage game payoffs

(2) "Payoffs need to be individually rational"
$\pi^{(i)} \geq 1$

Definition 3.21 (Individually rational payoff)

(1) For a given game $\Gamma = (N, \mathcal{A}, u)$ define player i's mini-max payoff as

$$\underline{u}^{(i)} = \min_{\sigma^{(-i)}} \max_{s^{(i)}} u^{(i)}(s^{(i)}, \sigma^{(-i)})$$

⇝ "worst-case scenario"

(2) A repeated game payoff $\pi^{(i)}$ is called individually rational if $\pi^{(i)} \geq \underline{u}^{(i)}$

Examples 3.22 (Mini-max)

(1) Prisoner's dilemma

	C	D
C	3,3	0,4
D	4,0	1,1

$\sigma^{(2)} = (x, 1-x)$

$u^{(1)}(C, \sigma^{(2)}) = 3x + 0 \cdot (1-x)$

$u^{(1)}(D, \sigma^{(2)}) = 4x + 1 - x = 1 + 3x$

$\min_{\sigma^{(-i)}} \max_{s^{(i)}} u^{(i)} = 1$

(2) In the definition of mini-max, it is important to allow for mixed strategies

	L	R
U	0.8, 0.4	0.4, 0.8
D	0.4, 0.8	0.8, 0.4

Claim: $\min_{s^{(2)}} \max_{s^{(1)}} u^{(1)}(s^{(1)}, s^{(2)}) = 0.8$

$\min_{\sigma^{(2)}} \max_{s^{(1)}} u^{(1)}(s^{(1)}, \sigma^{(2)}) = 0.6$

} Exercise

(3) The mini-max payoff can be below the Nash equilibrium payoff

	L	R
U	-2,2	1,-2
M	1,-2	-2,2
D	0,1	0,1

Claim: In any Nash equilibrium $\pi^{(1)} = 0$, $\pi^{(2)} = 1$

mini-max $\underline{u}^{(1)} = 0$ $\underline{u}^{(2)} = 0$

} Exercise

Theorem 3.23 (Folk theorem of repeated games)

For every feasible payoff vector $\pi = (\pi^{(1)}, \dots, \pi^{(n)})$ that is individually rational, one can find some $\hat{\delta} < 1$ such that for all $\delta > \hat{\delta}$ there is a NE with payoffs π.

"If players are sufficiently patient, almost anything goes"

Sketch of the proof:

(*) π is feasible ⇝ there is a sequence of actions $a_0, a_1, a_2, \dots$ such that the resulting average payoff is π

(C,C)
(C,C)
(D,D)
(C,C)
(C,C)
(D,D)

$2/3 \cdot 3 + 1/3 \cdot 1$

(*) Idea is similar to Grim/Trigger:

Strategy: Play sequence $a_0, a_1, a_2, \dots$ as long as everyone else does.

As soon as somebody deviates minimax this player forever.

Strategy	C	C	D	C	C	D	C	C	D	$2/3 \cdot 3$
Strategy	C	C	D	C	C	D	C	C	D	$+ 1/3 \cdot 1$
										$= 2 + 1/3$

Mutant	C	C	D	D	D	D	D
Strategy	C	C	D	C	D	D	D

$\underline{4}$ $\underline{1}$

Remark 3.24 (On the Folk theorem)

(1) There are various versions of the Folk theorem.
In particular under slightly more stringent assumptions, one can achieve subgame perfection.

(2) In a sense, the Folk theorem says it is really hard to make predictions for repeated games, even if everyone is rational.

(3) Repeated games are important (Cold War, firms)
⇝ Robert Aumann Nobel prize