GAME THEORY #9

Reminder (Static games with incomplete information)

(-) Basic setup: Players can be of different types $\theta^{(i)} \in \Theta^{(i)}$

Strategy is a map $s^{(i)}: \Theta^{(i)} \to A^{(i)}$

Types are drawn from a joint probability distribution $F(\Theta)$, $\Theta = (\theta^{(1)}, \ldots, \theta^{(n)})$

Usually we assume types are drawn independently

(-) Solution concept: Bayesian Nash equilibrium (BNE)

A strategy profile $\hat{\sigma} = (\hat{\sigma}^{(1)}, \ldots, \hat{\sigma}^{(n)})$ is a BNE if for all players i and all possible types $\theta^{(i)}$

$$\mathbb{E}_{\theta^{(-i)}} \pi^{(i)}(\hat{\sigma}^{(i)}, \hat{\sigma}^{(-i)}, \theta) \geq \mathbb{E}_{\theta^{(-i)}} \pi^{(i)}(\sigma^{(i)}, \hat{\sigma}^{(-i)}, \theta) \qquad \forall \sigma^{(i)}$$

(-) Examples: (*) Auctions

(*) Volunteer's dilemma

(-) Interesting observation:

(*) VD with complete information: has mixed Nash equilibrium with rather counterintuitive properties:

(1) Probability of playing each action depended only on the cost of co-player

(2) Why randomize at all?

(*) VD with incomplete information: There are pure equilibria that to an outside look as if they are mixed.

	C	D
C	$1-c^{(1)}, 1-c^{(2)}$	$1-c^{(1)}, 1$
D	$1, 1-c^{(2)}$	$0, 0$

$\sigma^{(1)} = (x, 1-x)$

$\sigma^{(2)} = (y, 1-y)$

$\pi^{(1)}(C, \sigma^{(2)}) = 1 - c^{(1)}$

$\pi^{(1)}(D, \sigma^{(2)}) = y$

$\rightsquigarrow y = 1 - c^{(1)}$

$\rightsquigarrow x = 1 - c^{(2)}$

Remark 4.6 (Harsanyi's purification theorem)

Consider some normal form game $\Gamma = (N, A, u)$

Look at a (slight) perturbation of this game: let $\theta_s^{(i)}$ be a random variable with range $[-1, 1]$, and define payoffs of the perturbed game as $\pi^{(i)}(s) = u^{(i)}(s) + \varepsilon \theta_s^{(i)}$

Then any Nash equilibrium of the unperturbed game is the limit of pure Nash equilibria of the perturbed game as $\varepsilon \to 0$.

§4.2) Dynamic games with incomplete information

Remark 4.7 (Motivation)

(*) In static games, the only way how players can learn something about other players' types is if there is some correlation between types

(*) In dynamic games, players make decisions at different stages. My action in an early stage might partly reveal my type.

Examples: (-) Poker: My bidding behavior tells you something about my cards.

(-) Breaking the enigma code

$\rightsquigarrow$ Players should update their beliefs over time.

Example 4.8 (MPRS game)

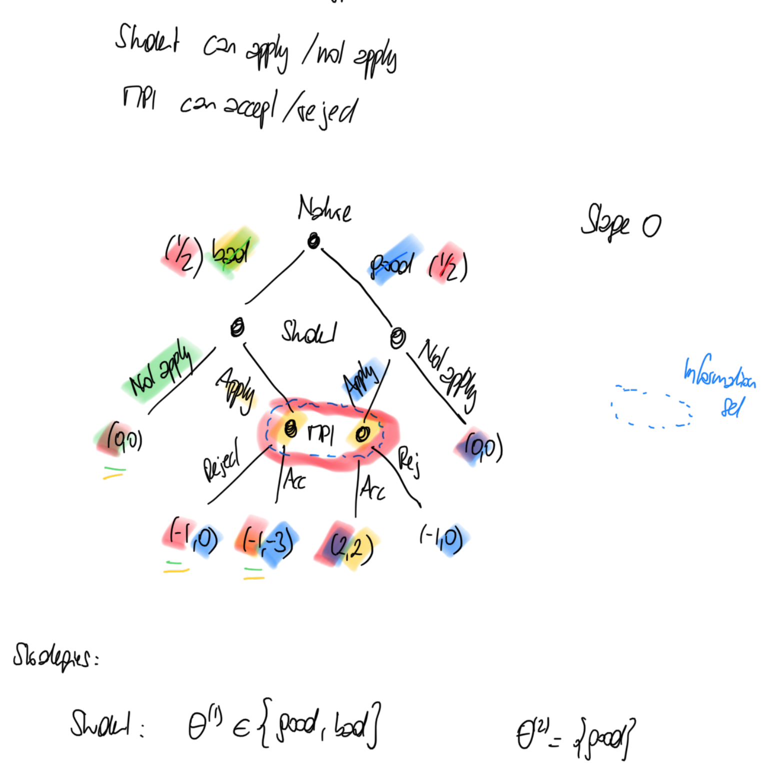
(*) Setup: There are students (player 1) and MPI (player 2)

Students can be either good or bad (50:50)

Student knows his/her type, MPI does not

Student can apply / not apply

MPI can accept / reject



(*) Strategies:

Student: $\theta^{(1)} \in \{good, bad\}$ $\qquad \Theta^{(2)} = \{good\}$

Strategies: $s^{(1)}: \Theta^{(1)} \to \{Apply, not\ apply\}$ $\qquad s^{(2)} \in \{Accept, Reject\}$

(*) Dream outcome for MPI: Only good students apply, those are accepted

$$s^{(1)} = \begin{cases} Apply & \text{if } \theta^{(1)} = good \\ Not\ apply & \text{if } \theta^{(1)} = bad \end{cases} \qquad s^{(2)} = Accept$$

Is this an equilibrium?

(-) Bad student: $\pi^{(1B)}(Apply, s^{(2)}) = -1$

$\pi^{(1B)}(Not\ apply, s^{(2)}) = 0$ ✓

(-) Good student $\pi^{(1G)}(Apply, s^{(2)}) = 2$ ✓

$\pi^{(1G)}(Not\ apply, s^{(2)}) = 0$

(-) MPI $\pi^{(2)}(Accept, s^{(1)}) = \underbrace{P(good \mid apply)}_{1} \cdot 2 + \underbrace{P(bad \mid apply)}_{0} \cdot (-3) = 2$ ✓

$\pi^{(2)}(Reject, s^{(1)}) = 0$

(*) Alternative equilibrium: $s^{(1)} = Not\ apply$ ✓ for all $\theta^{(1)}$

$s^{(2)} = Reject$ ✓

(-) Bad student $\pi^{(1B)}(Not\ apply, s^{(2)}) = 0$ ✓

$\pi^{(1B)}(Apply, s^{(2)}) = -1$

(-) Good student $\pi^{(1G)}(Not\ apply, s^{(2)}) = 0$ ✓

$\pi^{(1G)}(Apply, s^{(2)}) = -1$

(-) MPI $\pi^{(2)}(Accept, s^{(1)}) = P(good \mid apply) \cdot 2 + P(bad \mid apply) \cdot (-3) = -\frac{1}{2}$ $\quad 2$

($P(good \mid apply)$: $\frac{1}{2}$ / 1; $P(bad \mid apply)$: $\frac{1}{2}$ / 0) — Not an equilibrium

MPI needs to form "out of equilibrium beliefs"

Possible belief: $P(good \mid apply) = \frac{1}{2}$

$\pi^{(2)}(Reject, s^{(1)}) = 0$ ✓

Definition 4.9 (Signalling game)

In signalling games there are two players, a sender and a receiver. The type of the sender is private information.

First, the sender chooses an action.

Then, receiver observes sender's action and chooses her own action.

Definition 4.10 (Perfect Bayesian Nash equilibrium) PBNE

A PBNE is a strategy profile σ and (out of equilibrium) beliefs $\mu(\theta \mid a)$ such that

(1) Each player chooses an optimal strategy given the co-players' strategies and given beliefs

(2) $\mu(\theta \mid a)$ is updated with Bayes' rule whenever possible

$$\sum_\theta p(\theta) \cdot \sigma(a \mid \theta) > 0$$

1st equilibrium: $= p(good) s^{(1)}(Apply \mid good) + p(bad) s^{(1)}(Apply \mid bad) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2} > 0$

(3) If Bayesian updating is not possible, $\mu(\theta \mid a)$ can be an arbitrary probability distribution.

Example 4.11 (MPRS game revisited)

First equilibrium $\quad s^{(1)} = \begin{cases} Apply & \text{if } \theta^{(1)} = good \\ Not\ apply & \text{if } \theta^{(1)} = bad \end{cases}$

$s^{(2)} = Accept$

$\mu(good \mid Apply) = 1$

Second equilibrium $\quad s^{(1)} = Not\ apply \quad \forall \theta^{(1)}$

$s^{(2)} = Reject$

$\mu(good \mid Apply) = \frac{1}{2}$

Definition 4.12 (Types of equilibria)

(1) If in equilibrium all sender types use different actions $\rightsquigarrow$ "separating equilibrium" "revealing equilibrium"

$\rightsquigarrow$ After the game, everyone's type is known.

(2) All senders use the same action $\rightsquigarrow$ "pooling equilibrium"

(3) In between $\rightsquigarrow$ "partial pooling equilibrium" (Volunteer's dilemma)

Remark 4.13 (Equilibrium refinements)

The PBNE does not make any restrictions on which on which out-of-equilibrium beliefs are plausible.

As a result, there can be "too many" equilibria (from a modeler's perspective)

In the past 40 years, several suggestions have been made on how PBNE should be refined

"Intuitive criterion", "Divine criterion"

Idea: If receiver observes some action that should not happen in the first place, the receiver should assign zero probability that the action was sent by a sender who would never gain from this action.

Ex: Pooling equilibrium in MPRS game

$\mu(Bad \mid Apply) = 0 \Rightarrow \mu(Good \mid Apply) = 1$

$\Rightarrow$ MPI should accept

$\Rightarrow$ Good students should apply

Example 4.14 (Applications of signalling games)

(*) Peacock's tail ("Handicap principle")

(*) Stickers that indicate that you voted / donated blood } Moshe

(*) Third-party punishment } Moshe

(*) "Countersignalling"

(*) Job market signalling: Michael Spence (Nobel prize)

(*) Title and abstract of a paper

The end.